

Twisting functors for quantum group modules

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Abstract

We construct twisting functors for quantum group modules. First over the field $\mathbb{Q}(v)$ but later over any $\mathbb{Z}[v, v^{-1}]$ -algebra. The main results in this paper are a rigorous definition of these functors, a proof that they satisfy braid relations and applications to Verma modules.

Keywords: Quantum Groups; Quantized Enveloping Algebra; Twisting Functors; Representation Theory; Jantzen Filtration; Twisted Verma Modules

1 Introduction

Twisting functors were first introduced by S. Arkhipov (as a preprint in 2001 and published in [Ark04]). H. Andersen quantized the construction of twisting functors in [And03]. Each twisting functor T_w is defined via a so called semi-regular bimodule S_v^w . By the definition in [And03] its right module structure is not clear. Our first goal is to demonstrate that S_v^w is in fact a bimodule. We verify this by constructing an explicit isomorphism to an inductively defined right module. The calculations are in fact rather complicated and involve several manipulations with root vectors, see Section 2 below. At the same time these calculations will be essential in [Ped15a] and [Ped15b].

Once we have established the definition of the twisting functors we prove that they satisfy braid relations, see Proposition 3.11. In the ordinary (i.e. non-quantum) case the corresponding result was obtained by O. Khomenko and V. Mazorchuk in [KM05]. Our approach is similar but again the quantum case involves new difficulties, see Section 3. This section also contains an explicit proof of the fact that, for the longest word $w_0 \in W$, the twisting functor T_{w_0} takes a Verma module to its dual, see Theorem 3.9.

The above results have several applications in the representation theory of quantum group: They enable us to construct so called twisted Verma modules and Jantzen filtrations of (twisted) Verma modules with arbitrary (non-integral) weights and to derive the sum formula for these. In turn this simplifies the linkage principle in quantum category \mathcal{O}_q , q being a non-root of unity in an arbitrary field.

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1.2 Notation

In this paper we work with a quantum group over a semisimple Lie algebra \mathfrak{g} defined as in [Jan96]. Let Φ (resp. Φ^+ and Φ^-) denote the roots (resp. positive and negative roots) and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots. The quantum group has generators $\{E_\alpha, F_\alpha, K_\alpha | \alpha \in \Pi\}$ with relations as found in [Jan96]. Let $Q = \mathbb{Z}\Phi$ denote the root lattice. Let (a_{ij}) be the cartan matrix for \mathfrak{g} and let $(\cdot | \cdot)$ be the standard invariant bilinear form. Let $\Lambda = \text{span}_{\mathbb{Z}} \{\omega_1, \dots, \omega_n\} \subset \mathfrak{h}^*$ be the integral lattice where $\omega_i \in \mathfrak{h}^*$ is the fundamental weights defined by $(\omega_i | \alpha_j) = \delta_{ij}$. At first we work with the quantum group $U_v(\mathfrak{g})$ defined over $\mathbb{Q}(v)$ but later we will specialize to an arbitrary field and any nonzero q in the field. This is done by considering Lusztig's A -form U_A where $A = \mathbb{Z}[v, v^{-1}]$, see Section 4. For any A -algebra R ; $U_R = U_A \otimes_A R$. We will later need the automorphism ω of U_v and the antipode S defined as in [Jan96] along with the definition of quantum numbers $[n]_\beta$ and quantum binomial coefficients. We use the notation $E^{(r)} = \frac{E^r}{[r]!}$ and similarly for F . The Weyl group W is generated by the simple reflections $s_i = s_{\alpha_i}$. For a $w \in W$ let $l(w)$ be the length of W i.e. the smallest amount of simple reflections such that $w = s_{i_1} \cdots s_{i_{l(w)}}$. As usual we define for a weight $\mu \in \Lambda$ the weight space $(U_v)_\mu := \{u \in U_v | K_\alpha u = v^{(\alpha|\mu)} u \text{ for all } \alpha \in \Pi\}$. For a $\mu \in Q$, K_μ is defined as follows: $K_\mu = \prod_{i=1}^n K_{\alpha_i}^{a_i}$ if $\mu = \sum_{i=1}^n a_i \alpha_i$. There is a braid group action on the quantum group U_v usually denoted by T_{s_i} where s_i is the reflection with respect to the simple root α_i . In this paper we will reserve the T for twisting functors so we will call this braid group action R instead. That is we have automorphisms R_{s_i} such that

$$\begin{aligned} R_{s_i} E_{\alpha_i} &= -F_{\alpha_i} K_{\alpha_i} \\ R_{s_i} E_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v^{-s} E_{\alpha_i}^{(r)} E_{\alpha_j} E_{\alpha_i}^{(s)}, \text{ if } i \neq j \\ R_{s_i} F_{\alpha_i} &= -K_{\alpha_i}^{-1} E_{\alpha_i} \\ R_{s_i} F_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v^s F_{\alpha_i}^{(s)} F_{\alpha_j} F_{\alpha_i}^{(r)}, \text{ if } i \neq j \\ R_{s_i} K_\mu &= K_{s_i(\mu)}. \end{aligned}$$

Our definition of braid operators follows the definition in [Jan96]. Note that this definition differs slightly from the original definition in [Lus90] (cf. [Jan96, Warning 8.14]).

The inverse to R_{s_i} is given by

$$\begin{aligned} R_{s_i}^{-1} E_{\alpha_i} &= -K_{\alpha_i}^{-1} F_{\alpha_i} \\ R_{s_i}^{-1} E_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v^{-s} E_{\alpha_i}^{(s)} E_{\alpha_j} E_{\alpha_i}^{(r)}, \text{ if } i \neq j \\ R_{s_i}^{-1} F_{\alpha_i} &= -E_{\alpha_i} K_{\alpha_i} \\ R_{s_i}^{-1} F_{\alpha_j} &= \sum_{r+s=-a_{ij}} (-1)^s v^s F_{\alpha_i}^{(r)} F_{\alpha_j} F_{\alpha_i}^{(s)}, \text{ if } i \neq j \\ R_{s_i}^{-1} K_\mu &= K_{s_i(\mu)}. \end{aligned}$$

For $w \in W$ with a reduced expression $s_{i_1} \cdots s_{i_r}$, R_w is defined as $R_{s_{i_1}} \cdots R_{s_{i_r}}$. This is independent of the reduced expression of w . An important property of

the braid operators is that if $\alpha_{i_1}, \alpha_{i_2} \in \Pi$ and $w(\alpha_{i_1}) = \alpha_{i_2}$ then $R_w(F_{\alpha_{i_1}}) = F_{\alpha_{i_2}}$. These properties are proved in Chapter 8 in [Jan96].

For a reduced expression $s_{i_1} \cdots s_{i_N}$ of w_0 we can make an ordering of all the positive roots by defining

$$\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad j = 1, \dots, N$$

In this way we get $\{\beta_1, \dots, \beta_N\} = \Phi^+$. We could just as well have used the opposite reduced expression $w_0 = s_{i_N} \cdots s_{i_1}$. In the following we will sometimes use the numbering $s_{i_1} \cdots s_{i_N}$ and sometimes the numbering $s_{i_N} \cdots s_{i_1}$. Note that if $w = s_{i_1} \cdots s_{i_r}$ and we expand this to a reduced expression $s_{i_1} \cdots s_{i_r} s_{i_{r+1}} \cdots s_{i_N}$ we get $\{\beta_1, \dots, \beta_r\} = \Phi^+ \cap w(\Phi^-)$. We can define 'root vectors' $F_{\beta_j}, j = 1, \dots, N$ by

$$F_{\beta_j} := R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}}).$$

Note that this definition depends on the chosen reduced expression. For a different reduced expression we might get different root vectors. As mentioned above if $\beta \in \Pi$ then the root vector F_β defined above is the same as the generator with the same notation (cf. e.g [Jan96, Proposition 8.20]) so the notation is not ambiguous in this case. Let $w \in W$ and let $s_{i_r} \cdots s_{i_1}$ be a reduced expression of w . Define F_{β_j} by choosing a reduced expression $s_{i_1} \cdots s_{i_r} s_{i_{r+1}} \cdots s_{i_N}$ of w_0 starting with the reduced expression $s_{i_1} \cdots s_{i_r}$ of w^{-1} . We define a subspace $U_v^-(w)$ of U_v^- as follows:

$$U_v^-(w) := \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_j \in \mathbb{N} \right\}$$

where $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ as before. The definition of $U_v^-(w)$ seems to depend on the reduced expression of w . But the subspace is independent of the chosen reduced expression. This is shown in [Jan96, Proposition 8.22]. We will show below that $U_v^-(w)$ is a subalgebra of U_v^- and that

$$U_v^-(w) = \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \cdots F_{\beta_1}^{a_1} | a_j \in \mathbb{N} \right\}.$$

For a subalgebra $N \subset U_v$ we define $N^* = \bigoplus_\mu N_\mu^*$ (i.e. the graded dual) with the action given by $(uf)(x) = f(xu)$ for $u \in U_v, f \in N^*, x \in N$. We define 'the semiregular bimodule' $S_v^w := U_v \otimes_{U_v^-(w)} U_v^-(w)^*$. Proving that this is a U_v -bimodule will be the first main result of this paper. We will show that there exists a right module structure on S_v^w such that as a right module S_v^w is isomorphic to $U_v^-(w)^* \otimes_{U_v^-(w)} U_v$.

2 Calculations with root vectors

Let $A = \mathbb{Z}[v, v^{-1}]$. Lusztig's A -form is defined to be the A subalgebra of U_v generated by the divided powers $E_{\alpha_i}^{(n)}$ and $F_{\alpha_i}^{(n)}$ for $n \in \mathbb{N}$ and $K_i^{\pm 1}$.

We want to define $U_A^-(w) = \text{span}_A \left\{ F_{\beta_1}^{(a_1)} \cdots F_{\beta_r}^{(a_r)} | a_i \in \mathbb{N} \right\}$ where the F_{β_i} are defined from a reduced expression of w like earlier. We have $U_v^-(w_0) = U_v^-$ so we want a similar property over A : $U_A^-(w_0) = U_A^-$ where U_A^- is the A -subalgebra generated by $\{F_{\alpha_i}^{(n)} | n \in \mathbb{N}, i = 1, \dots, n\}$. This is shown very similar to the way it is shown for U_v in [Jan96].

Lemma 2.1 Assume \mathfrak{g} does not contain any G_2 components:

1. The subspace $U_A(w) := \text{span}_A \left\{ F_{\beta_1}^{(a_1)} \cdots F_{\beta_r}^{(a_r)} | a_i \in \mathbb{N} \right\}$ depends only on w , not on the reduced expression chosen for w .
2. Let α and β be two distinct simple roots. If w is the longest element in the subgroup of W generated by s_α and s_β then the span defined as before is the subalgebra of U_A generated by $F_\alpha^{(a)}$ and $F_\beta^{(b)}$, $a, b \in \mathbb{N}$.

Proof. Claim 2. is shown on a case by case basis. We will show first that the second claim implies the first.

We show this by induction on $l(w)$. If $l(w) \leq 1$ then there is only one reduced expression of w and there is nothing to show. Assume $l(w) > 1$ and that w has two reduced expressions $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}$ and $w = s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_r}$. We can assume that we can get from one of the reduced expression to the other by an elementary braid move ($s_\alpha s_\beta \cdots = s_\beta s_\alpha \cdots$). Set $\alpha = \alpha_1$ and $\gamma = \gamma_1$.

If $\alpha = \gamma$, set $w' = s_\alpha w$. Then the subspace spanned by the elements as in the lemma is for both expressions equal to:

$$\left(\sum_{a \geq 0} F_\alpha^{(a)} \right) \cdot R_{s_\alpha}(U_A^-(w')) \quad (1)$$

If $\alpha \neq \gamma$ then the elementary move must take place at the beginning of the reduced expression for both reduced expressions. Let w'' be the longest element generated by s_α and s_γ then we must have $w = w'' w'$ for some w' with $l(w'') + l(w') = l(w)$ and the reduced expression for w' in both reduced expressions are equal whereas the reduced expressions for w'' are the two possible combinations for the two different reduced expressions. So the span of the products is given by $U_A^-(w') R_{w''}(U_A^-(w''))$ which is independent of the reduced expression by the second claim.

We turn to the proof of the second claim: First assume we are in the simply laced case. Then $w = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$. Lets work with the reduced expression $s_\alpha s_\beta s_\alpha$. The other situation is symmetric by changing the role of α and β . We want to show that

$$B := \left\langle F_\alpha^{(n_1)}, F_\beta^{(n_2)} | n_1, n_2 \in \mathbb{N} \right\rangle_A = \text{span}_A \left\{ F_\alpha^{(a_1)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} | a_i \in \mathbb{N} \right\} =: V \quad (2)$$

where $F_{\alpha+\beta}^{(a)} = R_\alpha(F_\beta^{(a)})$. By [Lus90] section 5 we have that $F_{\alpha+\beta}^{(a)} \in U_A^-$ for all $a \in \mathbb{N}$ and we see that

$$F_\beta^{(k)} F_\alpha^{(k')} = \sum_{t, s \geq 0} (-1)^s v^{-tr-s} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_\beta^{(t)}$$

where the restrictions on the sum is $s+t = k'$ and $s+t = k$. Lusztig calculates for the E_α 's but just use the anti-automorphism Ω (defined in Section 1 of [Lus90]) on the results to get the corresponding formulas for the F 's. Also we get the $(-1)^s$ from the fact that (using the notation of [Lus90]) $E_{12} = -R_{\alpha_2}(E_{\alpha_1})$ because of the difference in the definition of the braid operators. Since $F_{\alpha+\beta}^{(a)} \in U_A^-$ we have that $V \subset B$. If we show that V is invariant by multiplication from

the left with $F_\alpha^{(a)}$ and $F_\beta^{(a)}$ for all $a \in \mathbb{N}$ then we must have $B \subset V$. For $F_\alpha^{(a)}$ this is clear. For $F_\beta^{(k)}$, $k \in \mathbb{N}$ we use the formula above:

$$\begin{aligned} F_\beta^{(k)} F_\alpha^{(a_1)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} &= \sum_{t,s \geq 0} (-1)^s v^{-d(tr+s)} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_\beta^{(t)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(a_3)} \\ &= \sum_{t,s \geq 0} (-1)^s v^{-d(tr+s)+dta_2} F_\alpha^{(r)} F_{\alpha+\beta}^{(s)} F_{\alpha+\beta}^{(a_2)} F_\beta^{(t)} F_\beta^{(a_3)} \\ &= \sum_{t,s \geq 0} (-1)^s v^{-d(tr+s)+dta_2} \begin{bmatrix} s+a_2 \\ s \end{bmatrix} \begin{bmatrix} t+a_3 \\ t \end{bmatrix} F_\alpha^{(r)} F_{\alpha+\beta}^{(s+a_2)} F_\beta^{(t+a_3)}. \end{aligned}$$

We see that $F_\beta^{(k)} V \subset V$ so $V = B$.

In the non simply laced case we have to use the formulas in [Lus90] section 5.3 (d)-(i) but the idea of the proof is the same. If there were similar formulas for the G_2 case it would be possible to show the same here. I do not know if similar formulas can be found in this case. The important part is just that if you 'v-commute' two of the 'root vectors' $F_{\beta_i}^{(k)}$ and $F_{\beta_j}^{(k')}$ you get something that is still in U_A . \square

Lemma 2.2

$$U_A^-(w_0) = U_A^-$$

Proof. It is clear that $U_A^-(w_0) \subset U_A^-$. We want to show that $F_\alpha^{(k)} U_A^-(w_0) \subset U_A^-(w_0)$ for all $\alpha \in \Pi$.

$U_A^-(w_0)$ is independent of the chosen reduced expression so we can choose a reduced expression for w_0 such that s_α is the last factor. Then the first root vector F_{β_1} is equal to F_α . Then it is clear that $F_\alpha^{(k)} U_A^-(w_0) \subset U_A^-(w_0)$. Since this was for an arbitrary simple root α the proof is finished. (This argument is sketched in the appendix of [Lus90].) \square

Corollary 2.3 *We get a basis of U_A^- by the products of the form $F_{\beta_1}^{(a_1)} \dots F_{\beta_N}^{(a_N)}$ where $a_1, \dots, a_N \in \mathbb{N}$.*

Corollary 2.4 $U_A^-(w) = U_v^-(w) \cap U_A$.

Proof. Assume the length of w is r and define for $k = (k_1, \dots, k_r) \in \mathbb{N}^r$

$$F^{(k)} = F_{\beta_1}^{(k_1)} \dots F_{\beta_r}^{(k_r)}.$$

It is clear that $U_A^-(w) \subseteq U_v^-(w) \cap U_A$. Assume $x \in U_v^-(w) \cap U_A$. Since $x \in U_v^-(w)$ we have constants $c_k \in \mathbb{Q}(v)$, $k \in \mathbb{N}^r$ such that

$$x = \sum_{k \in \mathbb{N}^r} c_k F^{(k)}.$$

Assume the length of w_0 is N and denote for $n \in \mathbb{N}^N$, $F^{(n)}$ like above for w . $U_v^-(w) \cap U_A \subseteq U_v^-(w_0) \cap U_A = U_A^-(w_0)$ ($U_A^-(w_0) \subset U_v^-(w_0) \cap U_A$ clearly and $U_A^-(w_0)$ is invariant under multiplication by U_A^- .) so there exists $b_n \in A$, $n \in \mathbb{N}^N$ such that

$$x = \sum_{k \in \mathbb{N}^N} b_k F^{(k)}.$$

But then we have two expressions of x in $U_v^-(w)$ expressed as a linear combination of basis elements. So we must have that the multindices b_k are zero on coordinates $\geq r$ and that all the c_k are actually in A . This proves the corollary. \square

Definition 2.5 Let $x \in (U_v)_\mu$ and $y \in (U_v)_\gamma$ then

$$[x, y]_v := xy - v^{-(\mu|\gamma)}yx.$$

Proposition 2.6 For $x_1 \in (U_v)_{\mu_1}$, $x_2 \in (U_v)_{\mu_2}$ and $y \in (U_v)_\gamma$ we have

$$[x_1x_2, y]_v = x_1[x_2, y]_v + v^{-(\gamma|\mu_2)}[x_1, y]_vx_2$$

and

$$[y, x_1x_2]_v = v^{-(\gamma|\mu_1)}x_1[y, x_2]_v + [y, x_1]_vx_2.$$

Proof. Direct calculation. \square

We have the following which corresponds to the Jacobi identity. Note that setting $v = 1$ recovers the usual Jacobi identity for the commutator.

Proposition 2.7 for $x \in (U_v)_\mu$, $y \in (U_v)_\nu$ and $z \in (U_v)_\gamma$ we have

$$[[x, y]_v, z]_v = [x, [y, z]_v]_v - v^{-(\mu|\nu)}[y, [x, z]_v]_v + v^{-(\nu|\mu+\gamma)}(v^{(\nu|\mu)} - v^{-(\nu|\mu)})[x, z]_vy$$

Proof. Direct calculation. \square

For use in the theorem below define:

Definition 2.8 Let $A = \mathbb{Z}[v, v^{-1}]$ and let A' be the localization of A in [2] (and/or [3]) if the Lie algebra contains any B_n, C_n or F_4 part (resp. any G_2 part). Let $w \in W$ have a reduced expression $s_{i_r} \cdots s_{i_1}$. Define β_j and F_{β_j} , $i = 1, \dots, r$ as above: $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$. We define

$$U_{A'}^-(w) = \text{span}_{A'} \left\{ F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_1, \dots, a_r \in \mathbb{N} \right\}$$

This subspace is independent of the reduced expression for w . This can be proved in the same way as Lemma 2.1 using the rank 2 calculations done in [Lus90].

The main tool that will be used in this project is the following theorem from [DP93, thm 9.3] originally from [LS91, Proposition 5.5.2]:

Theorem 2.9 Let F_{β_j} and F_{β_i} be defined as above. Let $i < j$. Let $A = \mathbb{Z}[v, v^{-1}]$ and let A' be the localization of A in [2] (and/or [3]) if the Lie algebra contains any B_n, C_n or F_4 part (resp. any G_2 part). Then

$$[F_{\beta_j}, F_{\beta_i}]_v = F_{\beta_j}F_{\beta_i} - v^{-(\beta_i|\beta_j)}F_{\beta_i}F_{\beta_j} \in \text{span}_{A'} \left\{ F_{\beta_{i+1}}^{a_{i+1}} \cdots F_{\beta_{j-1}}^{a_{j-1}} \right\}$$

Proof. We shall provide the details of the proof sketched in [DP93]. The rank 2 case is handled in [Lus90]. Note that in [Lus90] we see that when $\mu = 2$ (in his notation) we get second divided powers and when $\mu = 3$ we get third divided powers. This is one reason why we need to be able to divide by [2] and [3].

So we assume the rank 2 case is proven. In particular we can assume there is no G_2 component. Let $k \in \mathbb{N}$, $k < j$. Then $[F_{\beta_j}, F_{\beta_k}] = R_{s_{i_1}} \cdots R_{s_{i_{k-1}}} [R_{s_{i_k}} \cdots R_{s_{i_{j-1}}} (F_{\alpha_{i_j}}), F_{\alpha_{i_k}}]_v$ so we can assume in the above that $i = 1$. We can then assume that $j > 2$ because otherwise we would be in the rank 2 case. We will show by induction over $l \in \mathbb{N}$ that

$$[F_{\beta_t}, F_{\beta_1}]_v = F_{\beta_t} F_{\beta_1} - v^{-(\beta_1|\beta_t)} F_{\beta_1} F_{\beta_t} \in \text{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_{t-1}}^{a_{t-1}} \right\}$$

for all $1 < t \leq l$. The induction start $l = 2$ is the rank 2 case. Assume the induction hypothesis that

$$[F_{\beta_t}, F_{\beta_1}]_v = F_{\beta_t} F_{\beta_1} - v^{-(\beta_1|\beta_t)} F_{\beta_1} F_{\beta_t} \in \text{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_{t-1}}^{a_{t-1}} \right\}$$

for $t \leq l$. We need to prove the result for $l+1$. We have $\beta_{l+1} = s_{i_1} \cdots s_{i_l}(\alpha_{i_{l+1}})$. Now define $i = i_l$ and $j = i_{l+1}$. Set $w = s_{i_1} \cdots s_{i_{l-1}}$. So $\beta_{l+1} = w s_i(\alpha_j)$ and $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j})$. Define $\alpha = \alpha_{i_1}$. We need to show that

$$[R_w R_{s_i}(F_{\alpha_j}), F_{\alpha}]_v \in \text{span}_{A'} \left\{ F_{\beta_2}^{a_2} \cdots F_{\beta_l}^{a_l} \right\}.$$

We divide into cases:

Case 1) $(\alpha_i|\alpha_j) = 0$: In this case $R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_j})$. Since $s_i s_j = s_j s_i$ there is a reduced expression for w_0 starting with $s_{i_1} \cdots s_{l-1} s_j s_i$. So the induction hypothesis gives us that $[R_w(F_{\alpha_j}), F_{\alpha}]_v$ can be expressed by linear combinations of ordered monomials involving only $F_{\beta_2} \cdots F_{\beta_{l-1}}$.

Case 2) $(\alpha_i|\alpha_j) = -1$ and $l(ws_j) > l(w)$: In this case $ws_i s_j(\alpha_i) = w(\alpha_j) > 0$ so there is a reduced expression for w_0 starting with $s_{i_1} \cdots s_{i_{l-1}} s_i s_j s_i = s_{i_1} \cdots s_{i_{l-1}} s_j s_i s_j$. So we have by induction that $[R_w(F_{\alpha_j}), F_{\alpha}]_v$ is a linear combination of ordered monomials only involving $F_{\beta_2} \cdots F_{\beta_{l-1}}$.

Observe that we have

$$\begin{aligned} F_{\beta_{l+1}} &= R_w R_{s_i}(F_{\alpha_j}) \\ &= R_w(F_{\alpha_j} F_{\alpha_i} - v F_{\alpha_i} F_{\alpha_j}) \\ &= R_w(F_{\alpha_j}) F_{\beta_l} - v F_{\beta_l} R_w(F_{\alpha_j}) \\ &= [R_w(F_{\alpha_j}), F_{\beta_l}]_v \end{aligned}$$

so by Proposition 2.7 we get

$$\begin{aligned} [F_{\beta_{l+1}}, F_{\alpha}]_v &= [[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_{\alpha}]_v \\ &= [R_w(F_{\alpha_j}), [F_{\beta_l}, F_{\alpha}]_v]_v - v^{-(w(\alpha_j)|\beta_l)} [F_{\beta_l}, [R_w(F_{\alpha_j}), F_{\alpha}]_v]_v \\ &\quad + v^{-(\beta_l|\alpha+w(\alpha_j))} (v^{-1} - v) [R_w(F_{\alpha_j}), F_{\alpha}]_v F_{\beta_l}. \end{aligned}$$

By induction (and Proposition 2.6) $[R_w(F_{\alpha_j}), [F_{\beta_l}, F_{\alpha}]_v]_v$ and $[F_{\beta_l}, [R_w(F_{\alpha_j}), F_{\alpha}]_v]_v$ are linear combinations of ordered monomials containing only $F_{\beta_2}, \dots, F_{\beta_{l-1}}$ so we have proved this case.

Case 3) $\langle \alpha_i | \alpha_j \rangle = -1$ and $l(ws_j) < l(w)$: In this case write $u = ws_j$. We claim $l(us_i) > l(u)$. Assume $l(us_i) < l(u)$ then

$$l(w) + 2 = l(ws_i s_j) = l(us_j s_i s_j) = l(us_i s_j s_i) < l(u) + 2 = l(w) + 1$$

A contradiction. So there is a reduced expression of w_0 starting with us_i . We have $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_u(F_{\alpha_i})$ so we get

$$[F_{\beta_{l+1}}, F_{\alpha}]_v = [R_u(F_{\alpha_i}), F_{\alpha}]_v$$

Now we claim that either $u^{-1}(\alpha) = \alpha_j$ or $u^{-1}(\alpha) < 0$: Indeed $w^{-1}(\alpha) < 0$ so $u^{-1}(\alpha)$ is < 0 unless $w^{-1}(\alpha) = -\alpha_j$ in which case we get $u^{-1}(\alpha) = s_j w^{-1}(\alpha) = s_j(-\alpha_j) = \alpha_j$. If $\alpha = u(\alpha_j)$ we get

$$[R_u(F_{\alpha_i}), F_{\alpha}]_v = R_u([F_{\alpha_i}, F_{\alpha_j}]_v) = R_u(R_{s_j}(F_{\alpha_i})) = R_w(F_{\alpha_i}) = F_{\beta_l}$$

In the other case we know from induction that

$$[R_u(F_{\alpha_i}), F_{\alpha}]_v \in U_{A'}^-(u^{-1})$$

Now $U_{A'}^-(u^{-1}) \subset U_{A'}^-(s_j u^{-1}) = U_{A'}^-(w^{-1})$ so we get that $[R_u(F_{\alpha_i}), F_{\alpha}]_v$ can be expressed as a linear combination of monomials involving $F_{\alpha} = F_{\beta_1}$ and the terms $F_{\beta_2} \cdots F_{\beta_{l-1}}$. Assume that a monomial of the form $F_{\alpha}^m F_{\beta_2}^{a_2} \cdots F_{\beta_{l-1}}^{a_{l-1}}$ appears with nonzero coefficient. The weights of the left and right hand side must agree so we have $ws_i(\alpha_j) + \alpha = \sum_{k=2}^{l-1} a_k \beta_k + m\alpha$ or

$$ws_i(\alpha_j) = \sum_{k=2}^{l-1} a_k \beta_k + (m-1)\alpha$$

Since $w^{-1}(\beta_k) < 0$ for $k = 1, 2, \dots, l-1$ (and $\alpha = \beta_1$) we get

$$\alpha_i + \alpha_j = w^{-1}ws_i(\alpha_j) = \sum_{k=2}^{l-1} a_k w^{-1}(\beta_k) + (m-1)w^{-1}(\alpha) < 0.$$

Which is a contradiction.

Case 4) $\langle \alpha_j, \alpha_i^\vee \rangle = -1$, $\langle \alpha_i | \alpha_j \rangle = -2$ and $l(ws_j) > l(w)$: Here we get

$$F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}) = R_w(F_{\alpha_j}) F_{\beta_l} - v^2 F_{\beta_l} R_w(\alpha_j) = [R_w(F_{\alpha_j}), F_{\beta_l}]_v$$

From here the proof goes exactly as in case 2.

Case 5) $\langle \alpha_j, \alpha_i^\vee \rangle = -2$, and $l(ws_j) > l(w)$: First of all since $l(ws_j) > l(w)$ we can deduce that $l(ws_i s_j s_i s_j) = l(w) + 4$: We have $-\beta_{l+1} + 2ws_i s_j(\alpha_i) = ws_i s_j s_i(\alpha_j) = w(\alpha_j) > 0$ showing that we must have $ws_i s_j(\alpha_i) > 0$.

We have

$$F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_w(F_{\alpha_i} F_{\alpha_j}^{(2)} - v F_{\alpha_j} F_{\alpha_i} F_{\alpha_j} + v^2 F_{\alpha_j}^{(2)} F_{\alpha_i})$$

We claim that we have

$$R_{s_i}(F_{\alpha_j}) = \frac{1}{[2]} (R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\alpha_i} - F_{\alpha_i} R_{s_i} R_{s_j}(F_{\alpha_i}))$$

This is shown by a direct calculation. First note that

$$R_{s_i} R_{s_j}(F_{\alpha_i}) = R_{s_j}^{-1} R_{s_j} R_{s_i} R_{s_j}(F_{\alpha_i}) = R_{s_j}^{-1}(F_{\alpha_i}) = F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}$$

So

$$\begin{aligned} R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\alpha_i} - F_{\alpha_i} R_{s_i} R_{s_j}(F_{\alpha_i}) &= F_{\alpha_j} F_{\alpha_i}^2 - v^2 F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} - F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + v^2 F_{\alpha_i}^2 F_{\alpha_j} \\ &= F_{\alpha_j} F_{\alpha_i}^2 - v[2] F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + v^2 F_{\alpha_i}^2 F_{\alpha_j} \\ &= [2] R_{s_i}(F_{\alpha_i}). \end{aligned}$$

Therefore

$$\begin{aligned} F_{\beta_{l+1}} &= \frac{1}{[2]} (R_w R_{s_i} R_{s_j}(F_{\alpha_i}) F_{\beta_l} - F_{\beta_l} R_w R_{s_i} R_{s_j}(F_{\alpha_i})) \\ &= \frac{1}{[2]} [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\beta_l}]_v \\ &= \frac{1}{[2]} [[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_{\beta_l}]_v \end{aligned}$$

By Proposition 2.7 and the above we get

$$\begin{aligned} [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\alpha}]_v &= [[R_w(F_{\alpha_j}), F_{\beta_l}]_v, F_{\alpha}]_v \\ &= [R_w(F_{\alpha_j}), [F_{\beta_l}, F_{\alpha}]_v]_v - v^2 [F_{\beta_l}, [R_w(F_{\alpha_j}), F_{\alpha}]_v]_v \\ &\quad + v^{2-(\alpha|\beta_l)} (v^{-2} - v^2) [R_w(F_{\alpha_j}), F_{\alpha}]_v F_{\beta_l} \end{aligned}$$

which by induction is a linear combination of ordered monomials involving only $F_{\beta_2}, \dots, F_{\beta_l}$. Using Proposition 2.7 again we get

$$\begin{aligned} [2][F_{\beta_{l+1}}, F_{\alpha}]_v &= [[R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\beta_l}]_v, F_{\alpha}]_v \\ &= [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), [F_{\beta_l}, F_{\alpha}]_v]_v - [F_{\beta_l}, [R_w R_{s_i} R_{s_j}(F_{\alpha_i}), F_{\alpha}]_v]_v \end{aligned}$$

which by induction and the above is a linear combination of ordered monomials involving only $F_{\beta_2}, \dots, F_{\beta_l}$.

Case 6) $(\alpha_i|\alpha_j) = -2$, $l(ws_j) < l(w)$ and $l(ws_j s_i) < l(ws_j)$: Set $u = ws_j s_i$. We claim $l(us_i) = l(us_j) > l(u)$. Indeed suppose the contrary then $l(w) + 2 = l(ws_i s_j) = l(us_i s_j s_i s_j) < l(u) + 4 = l(w) + 2$. We reason like in case 3): We have $F_{\beta_{l+1}} = R_w R_{s_i}(F_{\alpha_j}) = R_u R_{s_i} R_{s_j} R_{s_i}(F_{\alpha_j}) = R_u(F_{\alpha_j})$. Now either $u^{-1}(\alpha) = \alpha_i$, $u^{-1}(\alpha) = s_i(\alpha_j)$ or $u^{-1}(\alpha) < 0$. If $u^{-1}(\alpha) < 0$ we get by induction that $[F_{\alpha}, R_u(F_{\alpha_j})]_v$ is in $U_{A'}^-(u^{-1}) \subset U_{A'}^-(w^{-1})$ and by essentially the same weight argument as in case 3) we are done.

If $\alpha = u(\alpha_i)$ then

$$\begin{aligned} [R_u(F_{\alpha_j}), F_{\alpha}]_v &= [R_u(F_{\alpha_j}), R_u(F_{\alpha_i})]_v \\ &= R_u(F_{\alpha_j} F_{\alpha_i} - v^2 F_{\alpha_i} F_{\alpha_j}) \\ &= \begin{cases} R_u R_{s_i}(F_{\alpha_j}) & \text{if } \langle \alpha_j, \alpha_i^\vee \rangle = -1 \\ R_u R_{s_i} R_{s_j}(F_{\alpha_i}) & \text{if } \langle \alpha_j, \alpha_i^\vee \rangle = -2 \end{cases} \end{aligned}$$

So $[F_{\alpha}, R_u(F_{\alpha_j})]_v \in U_{A'}^-(s_i s_j s_i u^{-1}) = U_{A'}^-(s_i w^{-1})$. Assume we have a monomial of the form $F_{\alpha}^m F_{\beta_2}^{a_2} \dots F_{\beta_l}^{a_l}$ with m nonzero in the expression of $[R_u(F_{\alpha_j}), F_{\alpha}]_v$.

Then

$$ws_i(\alpha_j) = \sum_{k=2}^l a_k \beta_k + (m-1)\alpha$$

and we get

$$\alpha_j = \sum_{k=2}^l a_k s_i w^{-1}(\beta_k) + (m-1)s_i w^{-1}(\alpha) < 0.$$

A contradiction.

If $\alpha = us_i(\alpha_j)$ then

$$\begin{aligned} [R_u(F_{\alpha_j}), F_{\alpha}]_v &= R_u[F_{\alpha_j}, R_{s_i}(F_{\alpha_j})]_v \\ &= R_u(F_{\alpha_j} R_{s_i}(F_{\alpha_j}) - v^{-2} R_{s_i}(F_{\alpha_j}) F_{\alpha_j}) \\ &= R_u(R_{s_i} R_{s_j} R_{s_i}(F_{\alpha_j}) R_{s_i}(F_{\alpha_j}) - v^{-2} R_{s_i}(F_{\alpha_j}) R_{s_i} R_{s_j} R_{s_i}(F_{\alpha_j})) \end{aligned}$$

Which is in $U_{A'}^-(s_i s_j s_i u^{-1}) = U_{A'}^-(s_i w^{-1})$ by the rank 2 case. By the same weight argument as above we are done.

Case 7) $(\alpha_i | \alpha_j) = -2$, $l(ws_j) < l(w)$ and $l(ws_j s_i) > l(ws_j)$: Set $u = ws_j$. Like in case 3) we get that either $u^{-1}(\alpha) = \alpha_j$ or $u^{-1}(\alpha) < 0$. If $\alpha = u(\alpha_j)$:

$$[F_{\beta_{l+1}}, F_{\alpha}]_v = R_u[R_{s_j} R_{s_i}(F_{\alpha_j}), F_{\alpha_j}]_v \in U_{A'}^-(s_i s_j u^{-1}) = U_{A'}^-(s_i w^{-1})$$

And by a weight argument as above we are done.

If $u^{-1}(\alpha) < 0$ then $\alpha = \beta'_i$ for some $i \in \{1, \dots, l-2\}$ where the β'_i 's are defined as above but using a reduced expression of u . Set $\beta'_{l-1} = u(\alpha_j)$, $\beta'_l = us_j(\alpha_i)$ and $\beta'_{l+1} = us_j s_i(\alpha_j) = ws_i(\alpha_j) = \beta_{l+1}$. Then

$$[F_{\beta_{l+1}}, F_{\alpha}]_v = [F_{\beta'_{l+1}}, F_{\beta'_i}]_v \in U_{A'}^-(s_i s_j u^{-1}) = U_{A'}^-(s_i w^{-1})$$

by induction and by a weight argument as above we are done. \square

Lemma 2.10 *Let $w_0 = s_{i_1} \cdots s_{i_N}$ and let $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ let $l, r \in \{1, \dots, N\}$ with $l \leq r$. Then*

$$\text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \cdots F_{\beta_l}^{a_l} | a_j \in \mathbb{N} \right\} = \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_l}^{a_l} \cdots F_{\beta_r}^{a_r} | a_j \in \mathbb{N} \right\}$$

and the subspace is invariant under multiplication from the left by F_{β_i} , $i = l, \dots, r$.

Proof. If $r - l = 0$ the lemma obviously holds. Assume $r - l > 0$. For $k \in \mathbb{N}^{r-l}$, $k = (k_l, \dots, k_r)$ let $F^k = F_{\beta_l}^{k_l} \cdots F_{\beta_r}^{k_r}$. We will prove the statement that $F^k \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \cdots F_{\beta_l}^{a_l} | a_j \in \mathbb{N} \right\}$ by induction over $k_l + \cdots + k_r$. If $k = 0$ the statement holds. We have

$$F^k = F_{\beta_j} F_{\beta_j}^{k_j-1} F_{\beta_{j+1}}^{k_{j+1}} \cdots F_{\beta_r}^{k_r}.$$

By induction $F_{\beta_j}^{k_j-1} F_{\beta_{j+1}}^{k_{j+1}} \cdots F_{\beta_r}^{k_r} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \cdots F_{\beta_l}^{a_l} | a_j \in \mathbb{N} \right\}$ so if we show that $F_{\beta_j} F_{\beta_r}^{b_r} \cdots F_{\beta_l}^{b_l} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \cdots F_{\beta_l}^{a_l} | a_i \in \mathbb{N} \right\}$ for all b_i , $i = l, \dots, r$ then we have shown the first inclusion.

We use downwards induction on j and induction on $b_1 + \dots + b_r$. If $j = r$ then this is obviously true. If $j < r$ we use theorem 2.9 to conclude that

$$F_{\beta_r} F_{\beta_j} - v^{-(\beta_r|\beta_j)} F_{\beta_j} F_{\beta_r} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_{r-1}}^{a_{r-1}} \dots F_{\beta_{j+1}}^{a_{j+1}} | a_i \in \mathbb{N} \right\}$$

If $b_r = 0$ the induction over j finishes the claim. We get now if $b_r \neq 0$

$$F_{\beta_j} F_{\beta_r}^{b_r} \dots F_{\beta_l}^{b_l} = v^{(\beta_r|\beta_j)} \left(F_{\beta_r} F_{\beta_j} F_{\beta_r}^{b_r-1} \dots F_{\beta_l}^{b_l} + \Sigma F_{\beta_r}^{b_r-1} \dots F_{\beta_l}^{b_l} \right)$$

where $\Sigma \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_{r-1}}^{a_{r-1}} \dots F_{\beta_{j+1}}^{a_{j+1}} | a_i \in \mathbb{N} \right\}$. By the induction on $b_r + \dots + b_l$ $F_{\beta_j} F_{\beta_r}^{b_r-1} \dots F_{\beta_l}^{b_l} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \dots F_{\beta_l}^{a_l} | a_i \in \mathbb{N} \right\}$ and the induction on j ensures that $\Sigma F_{\beta_r}^{b_r-1} \dots F_{\beta_l}^{b_l} \in \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \dots F_{\beta_l}^{a_l} | a_i \in \mathbb{N} \right\}$ since Σ contains only elements generated by $F_{\beta_{r-1}} \dots F_{\beta_l}$.

We have now shown that

$$\text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_l}^{a_l} \dots F_{\beta_r}^{a_r} | a_j \in \mathbb{N} \right\} \subset \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_r}^{a_r} \dots F_{\beta_l}^{a_l} | a_j \in \mathbb{N} \right\}$$

The other inclusion is shown symmetrically. In the process we also proved that the subspace is invariant under left multiplication by F_{β_j} . \square

Remark The above lemma shows that $U_v^-(w)$ is an algebra.

Definition 2.11 Let $\beta \in \Phi^+$ and let F_β be a root vector corresponding to β . Let $u \in U_q$. Define $\text{ad}(F_\beta^i)(u) := [[\dots[u, F_\beta]_v \dots]_v, F_\beta]_v$ and $\widetilde{\text{ad}}(F_\beta^i)(u) := [F_\beta, [\dots[F_\beta, u]_v \dots]]_v$ where the ' v -commutator' is taken i times from the left and right respectively.

Proposition 2.12 Let $u \in (U_A)_\mu$, $\beta \in \Phi^+$ and F_β a corresponding root vector. Set $r = \langle \mu, \beta^\vee \rangle$. Then in U_A we have the identity

$$\text{ad}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)}$$

and

$$\widetilde{\text{ad}}(F_\beta^i)(u) = [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(i-n)} u F_\beta^{(n)}$$

Proof. This is proved by induction. For $i = 0$ this is clear. The induction step

for the first claim:

$$\begin{aligned}
& [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)} F_\beta \\
& - v_\beta^{-r-2i} F_\beta [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)} F_\beta^{(n)} u F_\beta^{(i-n)} \\
& = [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)} [i+1-n] F_\beta^{(n)} u F_\beta^{(i+1-n)} \\
& - [i]_\beta! \sum_{n=0}^i (-1)^n v_\beta^{n(1-i-r)-r-2i} [n+1] F_\beta^{(n+1)} u F_\beta^{(i-n)} \\
& = [i]_\beta! \sum_{n=0}^{i+1} (-1)^n v_\beta^{n(-i-r)} \left(v_\beta^n [i+1-n] + v_\beta^{n-i-1} [n] \right) F_\beta^{(n)} u F_\beta^{(i+1-n)} \\
& = [i+1]_\beta! \sum_{n=0}^{i+1} (-1)^n v_\beta^{n(-i-r)} F_\beta^{(n)} u F_\beta^{(i+1-n)}.
\end{aligned}$$

The other claim is shown similarly by induction. \square

So we can define $\text{ad}(F_\beta^{(i)})(u) := ([i]!)^{-1} \text{ad}(F_\beta^i)(u) \in U_A$ and $\widetilde{\text{ad}}(F_\beta^{(i)})(u) := ([i]!)^{-1} \widetilde{\text{ad}}(F_\beta^i)(u) \in U_A$.

Proposition 2.13 *Let $a \in \mathbb{N}$, $u \in (U_A)_\mu$ and $r = \langle \mu, \beta^\vee \rangle$. In U_A we have the identities*

$$\begin{aligned}
u F_\beta^{(a)} &= \sum_{i=0}^a v_\beta^{(i-a)(r+i)} F_\beta^{(a-i)} \text{ad}(F_\beta^{(i)})(u) \\
&= \sum_{i=0}^a (-1)^i v_\beta^{a(r+i)-i} F_\beta^{(a-i)} \widetilde{\text{ad}}(F_\beta^{(i)})(u)
\end{aligned}$$

and

$$\begin{aligned}
F_\beta^{(a)} u &= \sum_{i=0}^a v_\beta^{(i-a)(r+i)} \widetilde{\text{ad}}(F_\beta^{(i)})(u) F_\beta^{(a-i)} \\
&= \sum_{i=0}^a (-1)^i v_\beta^{a(r+i)-i} \text{ad}(F_\beta^{(i)})(u) F_\beta^{(a-i)}
\end{aligned}$$

Proof. This is proved by induction. For $a = 0$ this is obvious. The induction

step for the first claim:

$$\begin{aligned}
[a+1]_\beta u F_\beta^{(a+1)} &= u F_\beta^{(a)} F_\beta \\
&= \sum_{i=0}^a v_\beta^{(i-a)(r+i)} F_\beta^{(a-i)} \operatorname{ad}(F_\beta^{(i)})(u) F_\beta \\
&= \sum_{i=0}^a v_\beta^{(i-a)(r+i)-r-2i} [a+1-i]_\beta F_\beta^{(a+1-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\
&\quad + \sum_{i=0}^a v_\beta^{(i-a)(r+i)} [i+1]_\beta F_\beta^{(a-i)} \operatorname{ad}(F_\beta^{(i+1)})(u) \\
&= \sum_{i=0}^a v_\beta^{(i-a-1)(r+i)-i} [a+1-i]_\beta F_\beta^{(a+1-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\
&\quad + \sum_{i=1}^{a+1} v_\beta^{(i-a-1)(r+i-1)} [i]_\beta F_\beta^{(a+1-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\
&= \sum_{i=0}^{a+1} v_\beta^{(i-a-1)(r+i)} \left(v_\beta^{-i} [a+1-i]_\beta + v_\beta^{a+1-i} [i] \right) F_\beta^{(a+1-i)} \operatorname{ad}(F_\beta^{(i)})(u) \\
&= [a+1]_\beta \sum_{i=0}^{a+1} v_\beta^{(i-a-1)(r+i)} F_\beta^{(a+1-i)} \operatorname{ad}(F_\beta^{(i)})(u).
\end{aligned}$$

So the induction step for the first identity is done. The three other identities are shown similarly by induction. \square

Let $s_{i_1} \dots s_{i_N}$ be a reduced expression of w_0 and construct root vectors F_{β_i} , $i = 1, \dots, N$. In the rest of the section F_{β_i} refers to the root vectors constructed as such. In particular we have an ordering of the root vectors.

Proposition 2.14 *Let $1 \leq i < j \leq N$ and $a, b \in \mathbb{Z}_{>0}$.*

$$[F_{\beta_j}^b, F_{\beta_i}^a]_v \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_i}^{a_i} \dots F_{\beta_j}^{a_j} \mid a_l \in \mathbb{N}, a_i < a, a_j < b \right\}.$$

Proof. From Theorem 2.9 we get the $a = 1, b = 1$ case. We will prove the general case by 2 inductions.

If $j - i = 1$ then $[F_{\beta_j}, F_{\beta_i}^a]_v = 0$ for all a . We will use induction over $j - i$.

We have by Proposition 2.6 that

$$[F_{\beta_j}, F_{\beta_i}^a]_v = v^{-(a-1)\beta_i|\beta_j)} F_{\beta_i}^{a-1} [F_{\beta_j}, F_{\beta_i}]_v + [F_{\beta_j}, F_{\beta_i}]_v F_{\beta_i}^{a-1}.$$

The first term is in the correct subspace by Theorem 2.9. On the second we use the fact that $[F_{\beta_i}, F_{\beta_j}]_v$ only contains factors $F_{\beta_{i+1}}^{a_i} \dots F_{\beta_{j-1}}^{a_{j-1}}$ and the induction over $j - i$ as well as induction over a to conclude that we can commute the $F_{\beta_i}^{a-1}$ to the correct place and be in the correct subspace.

Now just make a similar kind of induction on $i - j$ and b to get the result that

$$[F_{\beta_j}^b, F_{\beta_i}^a]_v \in \operatorname{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_i}^{a_i} \dots F_{\beta_j}^{a_j} \mid a_l \in \mathbb{N}, a_i < a, a_j < b \right\}. \quad \square$$

Corollary 2.15 *Let $1 \leq i < j \leq N$ and $a, b \in \mathbb{Z}_{>0}$.*

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v \in \text{span}_A \left\{ F_{\beta_i}^{(a_i)} \cdots F_{\beta_j}^{(a_j)} \mid a_i \in \mathbb{N}, a_i < a, a_j < b \right\}.$$

Proof. Proposition 2.14 tells us that there exists $c_k \in \mathbb{Q}(v)$ such that

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v = \sum_k c_k F_{\beta_i}^{(a_i^k)} \cdots F_{\beta_j}^{(a_j^k)}$$

with $a_i^k < a$ and $a_j^k < b$ for all k . But since $[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v \in U_A^-$ there exists $b_k \in A$ such that

$$[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v = \sum_k b_k F_{\beta_1}^{(a_1^k)} \cdots F_{\beta_N}^{(a_N^k)}.$$

Now we have two expressions of $[F_{\beta_j}^{(b)}, F_{\beta_i}^{(a)}]_v$ in terms of a basis of $U_{\mathbb{Q}(v)}^-$. So we must have that the c_k 's are equal to the b_k 's. Hence $c_k \in A$ for all k \square

Lemma 2.16 *Let $n \in \mathbb{N}$. Let $1 \leq j < k \leq N$.*

$\text{ad}(F_{\beta_j}^{(i)})(F_{\beta_k}^{(n)}) = 0$ and $\widetilde{\text{ad}}(F_{\beta_k}^{(i)})(F_{\beta_j}^{(n)}) = 0$ for $i \gg 0$.

Proof. We will prove the first assertion. The second is proved completely similar. We can assume $\beta_j = 1$ because

$$\text{ad}(F_{\beta_j}^{(i)})(F_{\beta_k}^{(n)}) = T_{s_{i_1}} \cdots T_{s_{i_{j-1}}} \left(\text{ad}(F_{\alpha_{i_j}}^{(i)})(T_{s_{i_j}} \cdots T_{s_{i_{k-1}}}(F_{\alpha_{i_k}}^{(n)})) \right).$$

So we assume $\beta_j = \beta_1 =: \beta \in \Pi$ and $\alpha := \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_j}) \in \Phi^+$. We have

$$\text{ad}(F_\beta)(F_\alpha^{(n)}) \in \text{span}_A \left\{ F_{\beta_2}^{(a_2)} \cdots F_{\beta_k}^{(a_k)} \mid a_l \in \mathbb{N}, a_k < n \right\},$$

hence the same must be true for $\text{ad}(F_\beta^{(i)})(F_\alpha^{(n)})$. By homogeneity if the monomial $F_{\beta_2}^{(a_2)} \cdots F_{\beta_k}^{(a_k)}$ appears with nonzero coefficient then we must have

$$i\beta + n\alpha = \sum_{s=2}^k a_s \beta_s$$

or equivalently

$$(n - a_k)\alpha = \sum_{s=2}^{k-1} a_s \beta_s - i\beta.$$

Use s_β on this to get

$$(n - a_k)s_\beta(\alpha) = \sum_{s=2}^{s-1} a_s s_\beta(\beta_s) + i\beta.$$

By the way the β_s 's are chosen $s_\beta(\beta_s) > 0$ for $1 < s < k$. So this implies that a positive multiple $(n - a_j)$ of a positive root must have $i\beta$ as coefficient. If we choose i greater than nd where d is the maximal possible coefficient of a simple root in any positive root then this is not possible. Hence we must have for $i > nd$ that $\text{ad}(F_\beta^{(i)})(F_\alpha^{(n)}) = 0$. \square

In the next lemma we will need to work with inverse powers of some of the F_β 's. We know from e.g. [And03] that $\{F_\alpha^a | a \in \mathbb{N}\}$, $\alpha \in \Pi$ is a multiplicative set so we can take the Ore localization in this set. Since R_w is an algebra isomorphism of U_v we can also take the Ore localization in one of the 'root vectors' F_{β_j} . We will denote the Ore localization in F_β by $U_{v(F_\beta)}$.

Lemma 2.17 *Let $\beta \in \Phi^+$ and F_β a root vector. Let $u \in (U_v)_\mu$ be such that $\widetilde{\text{ad}}(F_\beta^i)(u) = 0$ for $i \gg 0$. Let $a \in \mathbb{N}$ and set $r = \langle \mu, \beta^\vee \rangle$. Then in the algebra $U_{v(F_\beta)}$ we get*

$$uF_\beta^{-a} = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-i-a} \widetilde{\text{ad}}(F_\beta^i)(u)$$

and if $u' \in (U_v)_\mu$ is such that $\text{ad}(F_\beta^i)(u') = 0$ for $i \gg 0$

$$F_\beta^{-a}u' = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta \text{ad}(F_\beta^i)(u') F_\beta^{-i-a}.$$

Proof. First we want to show that

$$\widetilde{\text{ad}}(F_\beta^i)(u) F_\beta^{-1} = \sum_{k=i}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\text{ad}}(F_\beta^k)(u). \quad (3)$$

Remember that $\widetilde{\text{ad}}(F_\beta^k)(u) = 0$ for k big enough so this is a finite sum. This is shown by downwards induction on i . If i is big enough this is $0 = 0$. We have

$$F_\beta \widetilde{\text{ad}}(F_\beta^i)(u) = \widetilde{\text{ad}}(F_\beta^{i+1})(u) + v_\beta^{-r-2i} \widetilde{\text{ad}}(F_\beta^i)(u) F_\beta$$

so

$$\begin{aligned} \widetilde{\text{ad}}(F_\beta^i)(u) F_\beta^{-1} &= F_\beta^{-1} \widetilde{\text{ad}}(F_\beta^{i+1})(u) F_\beta^{-1} + v_\beta^{-r-2i} F_\beta^{-1} \widetilde{\text{ad}}(F_\beta^i)(u) \\ &= \sum_{k=i+1}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\text{ad}}(F_\beta^k)(u) + v_\beta^{-r-2i} F_\beta^{-1} \widetilde{\text{ad}}(F_\beta^i)(u) \\ &= \sum_{k=i}^{\infty} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\text{ad}}(F_\beta^k)(u). \end{aligned}$$

Setting $i = 0$ in the above we get the induction start:

$$uF_\beta^{-1} = \sum_{k \geq 0} v_\beta^{-r-2k} F_\beta^{-k-1} \widetilde{\text{ad}}(F_\beta^k)(u).$$

For the induction step assume

$$uF_\beta^{-a} = \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \widetilde{\text{ad}}(F_\beta^i)(u).$$

Then

$$\begin{aligned}
uF_\beta^{-a-1} &= \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \widetilde{\text{ad}}(F_\beta^i)(u) F_\beta^{-1} \\
&= \sum_{i \geq 0} v_\beta^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-i} \sum_{k \geq i} v_\beta^{-r-2k} F_\beta^{-k+i-1} \widetilde{\text{ad}}(F_\beta^k)(u) \\
&= \sum_{k \geq 0} \sum_{i=0}^k v_\beta^{-(a+1)r-(a+1)i-2k} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta F_\beta^{-a-1-k} \widetilde{\text{ad}}(F_\beta^k)(u) \\
&= \sum_{k \geq 0} v_\beta^{-(a+1)r-(a+2)k} \left(\sum_{i=0}^k v_\beta^{-(a+1)i+ak} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta \right) F_\beta^{-a-1-k} \widetilde{\text{ad}}(F_\beta^k)(u).
\end{aligned}$$

The induction is finished by observing that

$$\begin{aligned}
\sum_{i=0}^k v_\beta^{-(a+1)i+ak} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_\beta &= v_\beta^{ak} + \sum_{i=1}^k v_\beta^{-(a+1)i+ak} \left(v_\beta^i \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - v_\beta^{a+i} \begin{bmatrix} a+i-1 \\ i-1 \end{bmatrix}_\beta \right) \\
&= v_\beta^{ak} + \sum_{i=1}^k v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - \sum_{i=1}^k v_\beta^{-a(i-1)+ak} \begin{bmatrix} a+i-1 \\ i-1 \end{bmatrix}_\beta \\
&= v_\beta^{ak} + \sum_{i=1}^k v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta - \sum_{i=0}^{k-1} v_\beta^{-ai+ak} \begin{bmatrix} a+i \\ i \end{bmatrix}_\beta \\
&= \begin{bmatrix} a+k \\ k \end{bmatrix}_\beta.
\end{aligned}$$

The other identity is shown similarly by induction. \square

Definition 2.18 Let $\beta \in \Phi^+$ and let β be F_β a root vector. We define for $n \in \mathbb{N}$ in $U_{v(F_\beta)}$

$$F_\beta^{(-n)} = [n]! F_\beta^{-n}$$

$$\text{i.e. } F_\beta^{(-n)} = \left(F_\beta^{(n)} \right)^{-1}.$$

Corollary 2.19 Let $\beta \in \Phi^+$ and F_β a root vector. Let $u \in (U_v)_\mu$ be such that $\widetilde{\text{ad}}(F_\beta^{(i)})(u) = 0$ for $i \gg 0$. Let $a \in \mathbb{N}$ and set $r = \langle \mu, \beta^\vee \rangle$. Then in the algebra $U_{v(F_\beta)}$ we get

$$uF_\beta^{(-a)} F_\beta^{-1} = \sum_{i \geq 0} v_\beta^{-(a+1)r-(a+2)i} F_\beta^{(-i-a)} F_\beta^{-1} \widetilde{\text{ad}}(F_\beta^{(i)})(u)$$

and if $u' \in (U_v)_\mu$ is such that $\text{ad}(F_\beta^{(i)})(u') = 0$ for $i \gg 0$

$$F_\beta^{(-a)} F_\beta^{-1} u' = \sum_{i \geq 0} v_\beta^{-(a+1)r-(a+2)i} \text{ad}(F_\beta^{(i)})(u') F_\beta^{(-i-a)} F_\beta^{-1}.$$

3 Twisting functors

In this paper we are following the paper [And03] closely. The definition of twisting functors for quantum group modules given later and the ideas in this section are mostly coming from this paper.

We will start by showing that the semiregular bimodule S_v^w is a bimodule isomorphic to $U_v^-(w)^* \otimes_{U_v^-(w)} U_v$ as a right module.

Recall how $U_v(w)$, S_v^w and $S_v(F)$ are defined: Let $s_{i_r} \cdots s_{i_1}$ be a reduced expression for w and $F_{\beta_j} = R_{s_{i_1}} \cdots R_{s_{i_{j-1}}}(F_{\alpha_{i_j}})$ as usual then

$$U_v^-(w) = \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r} | a_i \in \mathbb{N} \right\},$$

$$S_v^w = U_v \otimes_{U_v^-(w)} U_v^-(w)^*$$

and for $F \in U_v^-$ such that $\{F^a | a \in \mathbb{N}\}$ is a multiplicative set

$$S_v(F) = U_{v(F)} / U_v$$

where $U_{v(F)}$ denotes the Ore localization in the multiplicative set $\{F^a | a \in \mathbb{N}\}$.

In the following proposition we will define a left U_v isomorphism between S_v^w and $S_v(F_{\beta_r}) \otimes_{U_v} S_v^{w'}$ where $w' = s_{i_r} w$. We will need some notation. Let $m \in \mathbb{N}$. We denote by $f_m^{(r)} \in (\mathbb{Q}(v)[F_{\beta_r}])^*$ the linear function defined by $f_m^{(r)}(F_{\beta_r}^a) = \delta_{m,a}$. We will drop the (r) from the notation in most of the following. For $g \in U_v^-(w')^*$ we define $f_m \cdot g$ to be the linear function defined by: For $x \in U_v^-(w')$, $(f_m \cdot g)(x F_{\beta_r}^a) = f_m(F_{\beta_r}^a) g(x)$. From the definition of $U_v^-(w)$ and because we are taking *graded* dual every $f \in U_v^-(w)^*$ is a linear combination of functions on the form $f_m \cdot g$ for some $m \in \mathbb{N}$ and $g \in U_v^-(w')$ (by induction this implies that every function in $U_v^-(w)$ is a linear combination of functions of the form $f_{m_r}^{(r)} \cdots f_{m_2}^{(2)} \cdot f_{m_1}^{(1)}$ for some $m_1, \dots, m_r \in \mathbb{N}$). Note that the definition of f_m makes sense for $m < 0$ but then $f_m = 0$.

Proposition 3.1 *Assume $w = s_{i_k} \cdots s_{i_1} = s_{i_k} w'$, where k is the length of w , then as a left U_v module*

$$S_v^w \cong S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'}$$

by the following left U_v isomorphism

$$\varphi_k : S_v^w \rightarrow S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'}$$

defined by:

$$\varphi_k(u \otimes f_m \cdot g) = u F_{\beta_k}^{-m-1} K_{\beta_k} \otimes (1 \otimes g), \quad u \in U_v, m \in \mathbb{N}, g \in U_v^-(w')^*.$$

The inverse to φ_k is the left U_v -homomorphism $\psi_k : S_v(F_{\beta_k}) \otimes_{U_v} S_v^{w'} \rightarrow S_v^w$ given by:

$$\psi_k(u F_{\beta_k}^{-m} \otimes (1 \otimes g)) = v^{(m\beta_k | \beta_k)} u K_{\beta_k}^{-1} \otimes f_{m-1} \cdot g, \quad u \in U_v, m \in \mathbb{N}, g \in U_v^-(w')^*.$$

Proof. The question is if φ_k is welldefined. Let $f = f_m \cdot g$. We need to show that the recipe for $u F_{\beta_j} \otimes f$ is the same as the recipe for $u \otimes F_{\beta_j} f$ for $j = 1, \dots, k$. For $j = k$ this is easy to see. Assume from now on that $j < k$. We need to

figure out what $F_{\beta_j} f$ is. We have by Proposition 2.13 (setting $r = \langle \beta_j, \beta_k^\vee \rangle$)

$$\begin{aligned}
(F_{\beta_j} f)(x F_{\beta_k}^a) &= f(x F_{\beta_k}^a F_{\beta_j}) \\
&= f \left(x \sum_{i=0}^a v_{\beta}^{(i-a)(r+i)} \begin{bmatrix} a \\ i \end{bmatrix}_{\beta} \widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) F_{\beta_k}^{a-i} \right) \\
&= \left(\sum_{i=0}^a v_{\beta}^{-m(r+i)} \begin{bmatrix} m+i \\ i \end{bmatrix}_{\beta} f_{m+i} \cdot \left(\widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) g \right) \right) (x F_{\beta_k}^a) \\
&= \left(\sum_{i \geq 0} v_{\beta}^{-m(r+i)} \begin{bmatrix} m+i \\ i \end{bmatrix}_{\beta} f_{m+i} \cdot \left(\widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) g \right) \right) (x F_{\beta_k}^a)
\end{aligned}$$

so

$$F_{\beta_j} f = \sum_{i \geq 0} v_{\beta}^{-m(r+i)} \begin{bmatrix} m+i \\ i \end{bmatrix}_{\beta} f_{m+i} \cdot \left(\widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) g \right).$$

Note that the sum is finite because of Lemma 2.16.

On the other hand we have that $u F_{\beta_j} \otimes f$ is sent to (using Lemma 2.17)

$$\begin{aligned}
&u F_{\beta_j} F_{\beta_k}^{-m-1} K_{\beta_k} \otimes (1 \otimes g) \\
&= u \sum_{i \geq 0} v_{\beta_k}^{-(m+1)r-(m+2)i} \begin{bmatrix} m+i \\ i \end{bmatrix}_{\beta} F_{\beta_k}^{-i-m-1} \widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) K_{\beta_k} \otimes (1 \otimes g) \\
&= u \sum_{i \geq 0} v_{\beta_k}^{-mr-mi} \begin{bmatrix} m+i \\ i \end{bmatrix}_{\beta} F_{\beta_k}^{-i-m-1} K_{\beta_k} \widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) \otimes (1 \otimes g).
\end{aligned}$$

Using the fact that $\widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j})$ can be moved over the first and the second tensor we see that the two expressions $u F_{\beta_j} \otimes f$ and $u \otimes F_{\beta_j} f$ are sent to the same.

So φ_k is a welldefined homomorphism. It is clear from the construction that φ_k is a U_v homomorphism.

We also need to prove that ψ_k is welldefined. We prove that $u F_{\beta_k}^{-m} F_{\beta_j} \otimes (1 \otimes g)$ is sent to the same as $u F_{\beta_k}^{-m} \otimes (1 \otimes F_{\beta_j} g)$ by induction over $k-j$. If $j = k-1$ we see from Lemma 2.17 and Theorem 2.9 that $F_{\beta_{k-1}} F_{\beta_k}^{-a} = v^{-(a\beta_k|\beta_{k-1})} F_{\beta_k}^{-a} F_{\beta_{k-1}}$ and therefore $u F_{\beta_k}^{-m} F_{\beta_{k-1}} \otimes (1 \otimes g)$ is sent to

$$\begin{aligned}
&v^{(m\beta_k-\beta_j|\beta_k)+(m\beta_k|\beta_{k-1})} u K_{\beta_k}^{-1} F_{\beta_{k-1}} \otimes f_{m-1} \cdot g \\
&= v^{(m\beta_k+(m-1)\beta_{k-1}|\beta_k)} u K_{\beta_k}^{-1} \otimes F_{\beta_{k-1}} (f_{m-1} \cdot g).
\end{aligned}$$

Note that because we have $\widetilde{\text{ad}}(F_{\beta_k}^i)(F_{\beta_j}) = 0$ for all $i \geq 1$ we get $F_{\beta_{k-1}} (f_{m-1} \cdot g) = v^{-(\beta_{k-1}|(m-1)\beta_k)} f_{m-1} \cdot (F_{\beta_{k-1}} g)$. Using this we see that $u F_{\beta_k}^{-m} F_{\beta_{k-1}} \otimes (1 \otimes g)$ is sent to the same as $u F_{\beta_k}^{-m} \otimes (1 \otimes F_{\beta_{k-1}} g)$.

Now assume $j - k > 1$. To calculate what $u F_{\beta_k}^{-m} F_{\beta_j} \otimes (1 \otimes g)$ is sent to we need to calculate $F_{\beta_k}^{-m} F_{\beta_j}$. By Lemma 2.17

$$F_{\beta_k}^{-m} F_{\beta_j} = v^{mr} F_{\beta_j} F_{\beta_k}^{-m} - \sum_{i \geq 1} v_{\beta}^{-(m+1)i} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_{\beta} F_{\beta_k}^{-m-i} \widetilde{\text{ad}}(F_{\beta_k}^i)(u).$$

So

$$uF_{\beta_k}^{-m}F_{\beta_j} \otimes (1 \otimes g) = u \left(v_{\beta}^{mr} F_{\beta_j} F_{\beta_k}^{-m} - \sum_{i \geq 1} v_{\beta}^{-(m+1)i} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_{\beta} F_{\beta_k}^{-m-i} \widetilde{\text{ad}}(F_{\beta_k}^i)(u) \right) \otimes (1 \otimes g).$$

By the induction over $k-j$ (remember that $\widetilde{\text{ad}}(F_{\beta_k}^i)(u)$ is a linear combination of ordered monomials involving only the elements $F_{\beta_{j+1}} \cdots F_{\beta_{k-1}}$) this is sent to the same as

$$u \left(v_{\beta}^{mr} F_{\beta_j} F_{\beta_k}^{-m} \otimes (1 \otimes g) - \sum_{i \geq 1} v_{\beta}^{-(m+1)i} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_{\beta} F_{\beta_k}^{-m-i} \otimes (1 \otimes \widetilde{\text{ad}}(F_{\beta_k}^i)(u)g) \right)$$

which is sent to

$$\begin{aligned} & u \left(v_{\beta}^{mr+2m} F_{\beta_j} K_{\beta_k}^{-1} \otimes f_{m-1} \cdot g - K_{\beta_k}^{-1} \otimes \sum_{i \geq 1} v_{\beta}^{2(m+i)-(m+1)i} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_{\beta} \otimes f_{m+i-1} \cdot (\widetilde{\text{ad}}(F_{\beta_k}^i)(u)g) \right) \\ &= v_{\beta}^{2m} u K_{\beta_k}^{-1} \left(v_{\beta}^{(m-1)r} F_{\beta_j} \otimes f_{m-1} \cdot g - 1 \otimes \sum_{i \geq 1} v_{\beta}^{-(m-1)i} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_{\beta} \otimes f_{m+i-1} \cdot (\widetilde{\text{ad}}(F_{\beta_k}^i)(u)g) \right) \\ &= v^{(m\beta_k|\beta_k)} u K_{\beta_k}^{-1} \otimes f_{m-1} \cdot (F_{\beta_j} g). \end{aligned}$$

But this is what $uF_{\beta_k}^{-m} \otimes (1 \otimes F_{\beta_j} g)$ is sent to. We have shown by induction that ψ_k is well defined. It is easy to check that ψ_k is the inverse to φ_k . \square

Proposition 3.2 *Let $s_{i_r} \cdots s_{i_1}$ be a reduced expression of $w \in W$. There exists an isomorphism of left U_v -modules*

$$S_v^w \cong S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1})$$

Proof. The proof is by induction of the length of w . Note that $S_v^e = U_v \otimes_k k^* \cong U_v$ so Proposition 3.1 with $w' = e$ gives the induction start.

Assume the length of w is $r > 1$. By Proposition 3.1 we have $S_v^w \cong S_v(F_{\beta_r}) \otimes_{U_v} S_v^{w'}$. By induction $S_v^{w'} \cong S_v(F_{\beta_{r-1}}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1})$. This finishes the proof. \square

We can now define a right action on S_v^w by the isomorphism in Proposition 3.2. By first glance this might depend on the chosen reduced expression for w . But the next proposition proves that this right action does not depend on the reduced expression chosen.

Proposition 3.3 *As a right U_v module $S_v^w \cong U_v^-(w)^* \otimes_{U_v} U_v$.*

Proof. All isomorphisms written in this proof are considered to be right U_v isomorphisms. This is proved in a very similar way to Proposition 3.1. We will sketch the proof here.

For $l \in \{1, \dots, N\}$ define $S_v^l = (U_v^l)^* \otimes_{U_v^l} U_v$ where $U_v^l = \text{span}_{\mathbb{Q}(v)} \left\{ F_{\beta_l}^{a_l} \cdots F_{\beta_r}^{a_r} \mid a_i \in \mathbb{N} \right\}$. Note that $S_v^1 = U_v^-(w)^* \otimes_{U_v} U_v$. We want to show that $(U_v^l)^* \otimes_{U_v^l} U_v \cong S_v^{l+1} \otimes_{U_v} S_v(F_{\beta_l})$. If we prove this we will have $S_v^1 \cong S_v^2 \otimes_{U_v} S_v(F_{\beta_1}) \cong \dots \cong S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong S_v^w$ as a right module and we are done.

Let $r = \langle \beta_j, \beta_l^\vee \rangle$. From Proposition 2.13 we have

$$F_{\beta_j} F_{\beta_l}^a = \sum_{i=0}^a v_{\beta}^{(i-a)(r+i)} \begin{bmatrix} a \\ i \end{bmatrix}_{\beta} F_{\beta_l}^{a-i} \text{ad}(F_{\beta_l}^i)(F_{\beta_j})$$

and by Lemma 2.17 we have

$$F_{\beta_l}^{-a} F_{\beta_j} = \sum_{i \geq 0} v_{\beta_l}^{-ar-(a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_{\beta_l} \text{ad}(F_{\beta_l}^i)(F_{\beta_r}) F_{\beta_l}^{-i-a}.$$

We define the right homomorphism φ_l from $(U_v^l)^* \otimes_{U_v^l} U_v$ to $S_v^{l+1} \otimes_{U_v} S_v(F_{\beta_l})$ by

$$\varphi_l(g \cdot f_{m_l} \otimes u) = (g \otimes 1) \otimes K_{\beta_l} F_{\beta_l}^{-m_l-1} u.$$

Like in the previous proposition we can use the above formulas to show that this is well defined and we can define an inverse like in the previous proposition only reversed. The inverse is:

$$\psi_l((g \otimes 1) \otimes F_{\beta_l}^{-m-1} u) = v^{-((m+1)\beta_l|\beta_l)} g \cdot f_m \otimes K_{\beta_l}^{-1} u. \quad \square$$

So we have now that S_v^w is a bimodule isomorphic to $U_v \otimes_{U_v^-(w)} U_v^-(w)^*$ as a left module and isomorphic to $U_v^-(w)^* \otimes_{U_v^-(w)} U_v$ as a right module. We want to examine the isomorphism between these two modules. For example what is the left action of K_{α} on $f \otimes 1 \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$.

Assume $f = f_{m_r}^{(r)} \cdots f_{m_1}^{(1)}$ i.e. that $f(F_{\beta_1}^{a_1} \cdots F_{\beta_r}^{a_r}) = \delta_{m_1, a_1} \cdots \delta_{m_r, a_r}$. Then we get via the isomorphism $(U_v^-(w))^* \otimes_{U_v^-(w)} U_v \cong S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1})$ that $f \otimes u$ is sent to

$$K_{\beta_r} F_{\beta_r}^{-m_r-1} \otimes \cdots \otimes K_{\beta_1} F_{\beta_1}^{-m_1-1} u.$$

We want to investigate what this is sent to under the isomorphism $S_v(F_{\beta_r}) \otimes_{U_v} \cdots \otimes_{U_v} S_v(F_{\beta_1}) \cong U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$. To do this we need to commute u with $F_{\beta_1}^{-m_1-1}$, then $F_{\beta_2}^{-m_2-1}$ and so on. So we need to find \tilde{u} and m'_1, \dots, m'_r such that

$$K_{\beta_r} F_{\beta_r}^{-m_r-1} \cdots K_{\beta_1} F_{\beta_1}^{-m_1-1} u = \tilde{u} K_{\beta_r} F_{\beta_r}^{-m'_r-1} \cdots K_{\beta_1} F_{\beta_1}^{-m'_1-1}$$

or equivalently

$$u F_{\beta_1}^{m'_1+1} K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m'_r+1} K_{\beta_r}^{-1} = F_{\beta_1}^{m_1+1} K_{\beta_1}^{-1} \cdots F_{\beta_r}^{m_r+1} K_{\beta_r}^{-1} \tilde{u}.$$

Assume we have found such \tilde{u} and m'_1, \dots, m'_r then the above tensor is sent to

$$v^{\sum_{i=1}^r ((m'_i+1)\beta_i|\beta_i)} \tilde{u} \otimes \tilde{f}$$

where $\tilde{f} = f_{m'_r}^{(r)} \cdots f_{m'_1}^{(1)}$. So in conclusion we have that $f \otimes u \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$ maps to $v^{\sum_{i=1}^r ((m'_i+1)\beta_i|\beta_i)} \tilde{u} \otimes \tilde{f} \in U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ where \tilde{f} and \tilde{u} are defined as above.

We have a similar result the other way: $u \otimes f \in U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ maps to $v^{-\sum_{i=1}^r ((m+1)\beta_i|\beta_i)} \bar{u} \otimes \bar{f} \in (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$. So if we want to figure out the left action of u on a tensor $f \otimes 1$ we need to first use the isomorphism $(U_v^-(w))^* \otimes_{U_v^-(w)} U_v \rightarrow U_v \otimes_{U_v^-(w)} (U_v^-(w))^*$ then use u on this and then use the isomorphism $U_v \otimes_{U_v^-(w)} (U_v^-(w))^* \rightarrow (U_v^-(w))^* \otimes_{U_v^-(w)} U_v$ back again.

In particular if $u = K_\alpha$ we have $\bar{f} = f$ and $\bar{u} = v^{\sum_{i=1}^r ((m_i+1)\beta_i|\beta_i)} K_\alpha$. Note that if $f = f_{m_r}^{(m_r)} \cdots f_{m_1}^{(1)}$ then the grading of f is $\sum_{i=1}^r m_i \beta_i$ so $K_\alpha(f \otimes 1) = v^{(\gamma + \sum_{i=1}^r \beta_i|\alpha)} f \otimes K_\alpha$ for $f \in (U_v^-(w))^*_\gamma$.

Definition 3.4 Let $w \in W$. For a U_v -module M define a 'twisted' version of M called ${}^w M$. The underlying space is M but the action on ${}^w M$ is given by: For $m \in M$ and $u \in U_v$

$$u \cdot m = R_{w^{-1}}(u)m.$$

Note that if $w, s \in W$ and $l(sw) > l(w)$ then ${}^s({}^w M) = {}^{sw} M$ since for $u \in U_v$ and $m \in {}^s({}^w M)$: $u \cdot m = R_s(u) \cdot m = R_{w^{-1}}(R_s(u))m = R_{(sw)^{-1}}(u)m$.

Definition 3.5 The twisting functor T_w associated to an element $w \in W$ is the following:

$T_w : U_v\text{-Mod} \rightarrow U_v\text{-Mod}$ is an endofunctor on $U_v\text{-Mod}$. For a U_v -module M :

$$T_w M = {}^w(S_v^w \otimes_{U_v} M).$$

Definition 3.6 Let M be a U_v -module and $\lambda : U_v^0 \rightarrow \mathbb{Q}(v)$ a character (i.e. an algebra homomorphism into $\mathbb{Q}(v)$). Then

$$M_\lambda = \{m \in M \mid \forall u \in U_v^0, um = \lambda(u)m\}.$$

Let X denote the set of characters. Let $\text{wt } M$ denote all the weights of M , i.e. $\text{wt } M = \{\lambda \in X \mid M_\lambda \neq 0\}$. We define for $\mu \in \Lambda$ the character v^μ by $v^\mu(K_\alpha) = v^{(\mu|\alpha)}$. We also define $v_\beta^\mu = v^{\frac{(\beta|\beta)}{2}\mu}$. We say that M only has integral weights if all its weights are of the form v^μ for some $\mu \in \Lambda$.

W acts on X by the following: For $\lambda \in X$ define $w\lambda$ by

$$(w\lambda)(u) = \lambda(R_{w^{-1}}(u)).$$

Note that $wv^\mu = v^{w(\mu)}$.

We will also need the dot action. It is defined as such: For a weight $\mu \in X$ and $w \in W$, $w \cdot \mu = v^{-\rho} w(v^\rho \mu)$ where $\rho = \frac{1}{2} \sum_{\beta \in \Phi} \beta$ as usual. The Verma module $M(\lambda)$ for $\lambda \in X$ is defined as $M(\lambda) = U_v \otimes_{U_v^{\geq 0}} \mathbb{Q}(v)_\lambda$ where $\mathbb{Q}(v)_\lambda$ is the one-dimensional module with trivial U_v^+ action and U_v^0 action by λ (i.e. $K_\mu \cdot 1 = \lambda(K_\mu)$). $M(\lambda)$ is a highest weight module generated by $v_\lambda = 1 \otimes 1$.

Note that $R_{w^{-1}}$ sends a weight space of weight μ to the weight space of weight $w(\mu)$ since if we have a vector m with weight μ in a module M we get in wM that

$$K_\alpha \cdot m = R_{w^{-1}}(K_\alpha)m = K_{w^{-1}(\alpha)}m = v^{(w^{-1}(\alpha)|\mu)}m = v^{(\alpha|w(\mu))}m.$$

We define the character of a U_v -module M as usual: The character is a map $\text{ch } M : X \rightarrow \mathbb{N}$ given by $\text{ch } M(\mu) = \dim M_\mu$. Let e^μ be the delta function $e^\mu(\gamma) = \delta_{\mu,\gamma}$. We will write $\text{ch } M$ as the formal infinite sum

$$\text{ch } M = \sum_{\mu \in X} \dim M_\mu e^\mu.$$

For more details see e.g. [Hum08]. Note that if we define $w(\sum_\mu a_\mu e^\mu) = \sum_\mu a_\mu e^{w(\mu)}$ then $\text{ch } {}^wM = w(\text{ch } M)$ by the above considerations.

Proposition 3.7

$$\text{ch } T_w M(\lambda) = \text{ch } M(w.\lambda)$$

Proof. To determine the character of $T_w M(\lambda)$ we would like to find a basis. We will do this by looking at some vector space isomorphisms to a space where we can easily find a basis. Then use the isomorphisms back again to determine what the basis looks like in $T_w M(\lambda)$. So assume $w = s_{i_r} \cdots s_{i_1}$ is a reduced expression for w . Expand to a reduced expression $s_{i_N} \cdots s_{i_{r+1}} s_{i_r} \cdots s_{i_1}$ for w_0 . Let $U_v^w = \text{span}_{\mathbb{Q}(v)} \{F_{\beta_{r+1}}^{a_{r+1}} \cdots F_{\beta_N}^{a_N} | a_i \in \mathbb{N}\}$. Set $k = \mathbb{Q}(v)$. We have the canonical vector space isomorphisms

$$\begin{aligned} U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v} U_v \otimes_{U_v^{\geq 0}} k_\lambda &\cong U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v^{\geq 0}} k_\lambda \\ &\cong U_v^-(w)^* \otimes_k U_v^w \otimes_k k_\lambda. \end{aligned}$$

The map from the last vector space to the first is easily seen to be $f \otimes u \otimes 1 \mapsto f \otimes u \otimes 1 \otimes 1 = f \otimes u \otimes v_\lambda$, $f \in U_v^-(w)^*$, $u \in U_v^w$ and $v_\lambda = 1 \otimes 1 \in U_v \otimes_{U_v^{\geq 0}} k_\lambda = M(\lambda)$ is a highest weight vector in $M(\lambda)$.

So we see that a basis of $T_w M(\lambda) = {}^w(U_v^-(w)^* \otimes_{U_v^-(w)} U_v \otimes_{U_v} M)$ is given by the following: Choose a basis $\{f_i\}_{i \in I}$ for $U_v^-(w)^*$ and a basis $\{u_j\}_{j \in J}$ for U_v^w . Then a basis for $T_w M(\lambda)$ is given by

$$\{f_i \otimes u_j \otimes v_\lambda\}_{i \in I, j \in J}.$$

So we can find the weights of $T_w M(\lambda)$ by examining the weights of $f \otimes u \otimes v_\lambda$ for $f \in U_v^-(w)^*$ and $u \in U_v^w$. By the remarks before this proposition we have that $K_\alpha(f \otimes 1) = v^{(\gamma + \sum_{i=1}^r \beta_i | \alpha)} f \otimes K_\alpha$ for $f \in U_v^-(w)^*_{v^\gamma}$ so for such f and for $u \in (U_v^w)_{v^\mu}$ the weight of $f \otimes u \otimes v_\lambda$ is $v^{\gamma + \mu + \sum_{i=1}^r \beta_i} \lambda$. After the twist with w the weight is $v^{w(\gamma + \mu)} w.\lambda$. The weights γ and μ are exactly such that $w(\gamma) < 0$ and $w(\mu) < 0$ so we see that the weights of $T_w M(\lambda)$ are $\{v^\mu w.\lambda | \mu < 0\}$ each with multiplicity $\mathcal{P}(\mu)$ where \mathcal{P} is Kostant's partition function. This proves that the character is the same as the character for the Verma module $M(w.\lambda)$. \square

Definition 3.8 Let $\lambda \in X$ and $M(\lambda)$ the Verma module with highest weight λ . Let $w \in W$. We define

$$M^w(\lambda) = T_w M(w^{-1}.\lambda).$$

Recall the duality functor $D : U_v - \text{Mod} \rightarrow U_v - \text{Mod}$. For a U_v module M , $DM = \text{Hom}(M, \mathbb{Q}(v))$ is the graded dual module with action given by $(xf)(m) = f(S(\omega(m)))$ for $x \in U_v$, $f \in DM$ and $m \in M$. By this definition we have $\text{ch } DM = \text{ch } M$ and $D(DM) = M$.

Theorem 3.9 *Let w_0 be the longest element in the Weyl group. Let $\lambda \in X$. Then*

$$T_{w_0}M(\lambda) \cong DM(w_0.\lambda)$$

Proof. We will show that $DT_{w_0}M(w_0.\lambda) \cong M(\lambda)$ by showing that $DT_{w_0}M(w_0.\lambda)$ is a highest weight module with highest weight λ . We already know that the characters are equal by Proposition 3.7 so all we need to show is that $DT_{w_0}M(w_0.\lambda)$ has a highest weight vector of weight λ that generates the whole module over U_v . Consider the function $g_\lambda \in DM^{w_0}(\lambda)$ given by:

$$g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = \begin{cases} 1 & \text{if } a_N = \dots = a_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}$ with $a_i \in \mathbb{N}$ defines a basis for $M^{w_0}(\lambda)$ so this defines a function on $M^{w_0}(\lambda)$. In the proof of Proposition 3.7 we see that a basis is given by $f \otimes 1 \otimes v_\lambda \in U_v^-(w_0) \otimes U_v \otimes M(\lambda) = T_{w_0}M(\lambda)$. We know that elements of the form $f_{m_N}^{(N)} \dots f_{m_1}^{(1)}$ defines a basis of $(U_v^-)^* = U_v^-(w_0)^*$. Under the isomorphisms of Proposition 3.3 $f_{m_N}^{(N)} \dots f_{m_1}^{(1)} \otimes 1 \otimes v_{w_0.\lambda}$ is sent to

$$K_{\beta_N} F_{\beta_N}^{-m_N-1} \otimes \dots \otimes K_{\beta_1} F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda} \in S_v(F_{\beta_N}) \otimes_{U_v} \dots \otimes_{U_v} S_v(F_{\beta_1}) \otimes_{U_v} M(w_0.\lambda).$$

If we commute all the K 's to the right to the v_λ we get some non-zero multiple of

$$F_{\beta_N}^{-m_N-1} \otimes \dots \otimes F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda}.$$

So we have shown that $\{F_{\beta_N}^{-m_N-1} \otimes \dots \otimes F_{\beta_1}^{-m_1-1} \otimes v_{w_0.\lambda} | m_i \in \mathbb{N}\}$ is a basis of $M^{w_0}(\lambda)$.

The action on a dual module DM is given by $uf(u') = f(S(\omega(u)u'))$. Remember that the action on $M^{w_0}(\lambda)$ is twisted by R_{w_0} so we get that

$$ug_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = g_\lambda(R_{w_0}(S(\omega(u)))F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}).$$

In particular for $u = K_\mu$ we get

$$\begin{aligned} K_\mu g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) &= g_\lambda(K_{w_0(\mu)} F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\ &= v^c(w_0.\lambda)(K_{w_0(\mu)})g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \end{aligned}$$

where

$$c = (w_0(\mu)) \left| \sum_{i=1}^N a_i \beta_i + \sum_{i=1}^N \beta_i \right|.$$

we have

$$\begin{aligned}
v^c(w_0.\lambda)(K_{w_0(\mu)}) &= v^{(w_0(\mu)|\sum_{i=1}^N a_i\beta_i + \sum_{i=1}^N \beta_i)}(v^{-\rho}w_0(v^\rho\lambda))(K_{w_0(\mu)}) \\
&= v^{(w_0(\mu)|\sum_{i=1}^N a_i\beta_i + 2\rho)}v^{-(\rho|w_0(\mu))}(v^\rho\lambda)(K_\mu) \\
&= v^{(w_0(\mu)|\sum_{i=1}^N a_i\beta_i + \rho)}v^{(\rho|\mu)}\lambda(K_\mu) \\
&= v^{(w_0(\mu)|\sum_{i=1}^N a_i\beta_i + \rho)}v^{-(\rho|w_0(\mu))}\lambda(K_\mu) \\
&= v^{(w_0(\mu)|\sum_{i=1}^N a_i\beta_i)}\lambda(K_\mu).
\end{aligned}$$

Setting the a_i 's equal to zero we get $\lambda(K_\mu)$. So g_λ has weight λ . We want to show that g_λ generates $DM^{w_0}(\lambda)$ over U_v .

Let $M \in \mathbb{N}^N$, $M = (m_1, \dots, m_N)$. An element in $DM^{w_0}(\lambda)$ is a linear combination of elements of the form g_M defined by:

$$g_M(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = \delta_{a_1, m_1} \cdots \delta_{a_N, m_N}.$$

This is because of the way the dual module is defined (as the graded dual). We want to show that $g_M \in U_v g_\lambda$ by using induction over $m_1 + \dots + m_N$. Note that $g_{(0, \dots, 0)} = g_\lambda$ so this gives the induction start. Assume $M = (m_1, \dots, m_N) \in \mathbb{N}^N$. Let j be such that $m_N = \dots = m_{j+1} = 0$ and $m_j > 0$. By induction we get for $M' = (0, \dots, 0, m_j - 1, m_{j-1}, \dots, m_1)$ that $g_{M'} \in U_v g_\lambda$. Now let $u_j = \omega(S^{-1}(R_{w_0}^{-1}(F_{\beta_j})))$. Then

$$u_j g_\lambda(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) = g_\lambda(F_{\beta_j} F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}).$$

From Lemma 2.17 we get for $r > j$ (setting $k = \langle \beta_j, \beta_r^\vee \rangle$)

$$F_{\beta_j} F_{\beta_r}^{-a} = v_{\beta_r}^{-ak} F_{\beta_r}^{-a} + \sum_{i \geq 1} v_{\beta_r}^{-ak - (a+1)i} \begin{bmatrix} a+i-1 \\ i \end{bmatrix}_{\beta_r} F_{\beta_r}^{-i-a} \widetilde{\text{ad}}(F_{\beta_r}^i)(u).$$

But $g_{M'}$ is zero on every $F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}$ where one of the a_i 's with $i > j$ is strictly greater than zero. This coupled with the observation above gives us that

$$\begin{aligned}
&u_j g_{M'}(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\
&= g_{M'}(v^c F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_j}^{-(a_j-1)-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda}) \\
&= v^c g_M(F_{\beta_N}^{-a_N-1} \otimes \dots \otimes F_{\beta_j}^{-a_j-1} \otimes \dots \otimes F_{\beta_1}^{-a_1-1} \otimes v_{w_0.\lambda})
\end{aligned}$$

where c is some constant coming from the commutations. We see that $g_M = v^{-c} u_j g_{M'}$ which finishes the induction step.

So in conclusion we have that $DM^{w_0}(\lambda)$ is a highest weight module with highest weight λ . So we have a surjection from $M(\lambda)$ to $DM^{w_0}(\lambda)$. But since the two modules have the same character and the weight spaces are finite dimensional the surjection must be an isomorphism. \square

Proposition 3.10 *Let M be a U_v -module, $\beta \in \Phi^+$ and let $w \in W$. Assume $s_{i_r} \cdots s_{i_1}$ is a reduced expression of w and $F_\beta = R_{s_{i_1}} \cdots R_{s_{i_r}}(F_\alpha)$ for some $\alpha \in \Pi$ such that $l(s_\alpha w) > l(w)$ (so we have $w(\beta) = \alpha$). Then*

$${}^w(S_v(F_\beta) \otimes_{U_v} M) \cong S_v(F_\alpha) \otimes_{U_v} {}^w M.$$

Proof. Define the map $\varphi : S_v(F_\alpha) \otimes {}^w M \rightarrow {}^w(S_v(F_\beta) \otimes M)$ by

$$\varphi(uF_\alpha^{-m} \otimes m) = R_{w^{-1}}(u)F_\beta^{-m} \otimes m.$$

This is obviously a U_v -homomorphism if it is welldefined and it is a bijection because $R_{w^{-1}}$ is a U_v -isomorphism. We have to check that if $uF_\alpha^{-m} = u'F_\alpha^{-m'}$ then $R_{w^{-1}}(u)F_\beta^{-m} = R_{w^{-1}}(u')F_\beta^{-m'}$ and that $\varphi(uF_\alpha^{-m}u' \otimes m) = \varphi(uF_\alpha^{-m} \otimes R_{w^{-1}}(u')m)$ but $uF_\alpha^{-m} = u'F_\alpha^{-m'}$ if and only if $F_\alpha^{m'}u = F_\alpha^m u'$. Using the isomorphism $R_{w^{-1}}$ on this we get $F_\beta^{m'}R_{w^{-1}}(u) = F_\beta^m R_{w^{-1}}(u')$ which implies $R_{w^{-1}}(u)F_\beta^{-m} = R_{w^{-1}}(u')F_\beta^{-m'}$. For the other equation: Since we only have the definition of φ on elements on the form $uF_\alpha^{-m} \otimes m$ assume $F_\alpha^{-m}u' = \tilde{u}F_\beta^{-\tilde{m}}$. This is equivalent to $u'F_\alpha^{\tilde{m}} = F_\alpha^m \tilde{u}$. Use $R_{w^{-1}}$ on this to get $R_{w^{-1}}(u')F_\beta^{\tilde{m}} = F_\beta^m \tilde{u}$ or equivalently $F_\beta^{-m}R_{w^{-1}}(u) = R_{w^{-1}}(\tilde{u})F_\alpha^{-\tilde{m}}$. Now we can calculate:

$$\begin{aligned} \varphi(uF_\alpha^{-m}u' \otimes m) &= \varphi(u\tilde{u}F_\alpha^{-\tilde{m}} \otimes m) \\ &= R_{w^{-1}}(u\tilde{u})F_\beta^{-\tilde{m}} \otimes m \\ &= R_{w^{-1}}(u)R_{w^{-1}}(\tilde{u})F_\beta^{-\tilde{m}} \otimes m \\ &= R_{w^{-1}}(u)F_\beta^{-m} \otimes R_{w^{-1}}(u')m = \varphi(uF_\alpha^{-m} \otimes R_{w^{-1}}(u')m). \quad \square \end{aligned}$$

Proposition 3.11 $w \in W$. If s is a simple reflection such that $sw > w$ then

$$T_{sw} = T_s \circ T_w.$$

Proof. Let α be the simple root corresponding to the simple reflection s . By Proposition 3.2 we get for M a U_v -module:

$$\begin{aligned} T_{sw}M &= {}^{sw}(S_v^{sw} \otimes_{U_v} M) \cong {}^{sw}(S_v(R_{w^{-1}}(F_\alpha)) \otimes_{U_v} S_v^w \otimes_{U_v} M) \\ &\cong {}^s({}^w(S_v(R_{w^{-1}}(F_\alpha)) \otimes_{U_v} S_v^w \otimes_{U_v} M)) \\ &\cong {}^s(S_v(F_\alpha) \otimes_{U_v} {}^w(S_v^w \otimes_{U_v} M)) \end{aligned}$$

where the last isomorphism is the one from Proposition 3.10. \square

4 Twisting functors over Lusztigs A-form

We want to define twisting functors so they make sense to apply to U_A modules. Note first that the maps R_s send U_A to U_A .

Recall that for $n \in \mathbb{N}$ with $n > 0$ and F_β a root vector we have defined in $U_{v(F_\beta)}$

$$F_\beta^{(-n)} = [n]_\beta! F_\beta^{-n} \tag{4}$$

$$\text{i.e. } F_\beta^{(-n)} = \left(F_\beta^{(n)}\right)^{-1}.$$

Definition 4.1 Let s be a simple reflection corresponding to a simple root α . Let S_A^s be the U_A -sub-bimodule of $S_v^s = S_v(F_\alpha)$ generated by the elements $\{F_\alpha^{(-n)}F_\alpha^{-1} | n \in \mathbb{N}\}$.

Note that $S_A^s \otimes_A \mathbb{Q}(v) = S_v^s$.

Proposition 4.2 *In $U_v(\mathfrak{sl}_2)$ let E, K, F be the usual generators and define as in [Lus90] the elements*

$$\begin{bmatrix} K; c \\ t \end{bmatrix} = \prod_{n=1}^t \frac{Kv^{c-n+1} - K^{-1}v^{-c+n-1}}{v^s - v^{-s}}.$$

Then

$$F^{(-s)}F^{-1}E^{(r)} = \sum_{t=0}^r E^{(r-t)} \begin{bmatrix} K; r-s-t-2 \\ t \end{bmatrix} F^{(-s-t)}F^{-1}.$$

Proof. This is proved by induction over r . We define as in [Jan96]

$$[K; c] = \begin{bmatrix} K; c \\ 1 \end{bmatrix} = \frac{Kv^c - K^{-1}v^{-c}}{v - v^{-1}}.$$

From [Jan96] we get $EF^{s+1} = F^{s+1}E + [s+1]F^s[K, -s]$ so

$$F^{-s-1}E = EF^{-s-1} + [s+1]F^{-1}[K; -s]F^{-s-1} = EF^{-s-1} + [s+1][K; -2-s]F^{-s-2}$$

and multiplying with $[s]!$ we get

$$F^{(-s)}F^{-1}E = EF^{(-s)}F^{-1} + [K; -2-s]F^{(-s-1)}F^{-1}.$$

This is the induction start. The rest is the induction step. In the process you have to use that

$$\frac{1}{[r]} \left([r-t] \begin{bmatrix} K; r-s-t \\ t \end{bmatrix} + \begin{bmatrix} K; r-1-s-t \\ t-1 \end{bmatrix} [K; -s-t] \right) = \begin{bmatrix} K; r-s-t-1 \\ t \end{bmatrix}$$

or equivalently that

$$[r-t][K; r-s-t] + [t][K; -s-t] = [r][K; r-s-2t].$$

This can be shown by a direct calculation. □

We could have proved this in the other way around instead too to get

Proposition 4.3

$$E^{(r)}F^{(-s)}F^{-1} = \sum_{t=0}^r F^{(-s-t)}F^{-1} \begin{bmatrix} K; s+t-r+2 \\ t \end{bmatrix} E^{(r-t)}.$$

The above and Corollary 2.19 shows that $S_A(F)$ is a bimodule. We can now define the twisting functor T_s^A corresponding to s :

Definition 4.4 *Let s be a simple reflection corresponding to a simple root α . The twisting functor $T_s^A : U_A\text{-Mod} \rightarrow U_A\text{-Mod}$ is defined by: Let M be a U_A module, then*

$$T_s^A(M) = {}^s(S_A(F_\alpha) \otimes_{U_A} M).$$

Note that $T_s^A(M) \otimes_A \mathbb{Q}(v) = T_s(M \otimes_A \mathbb{Q}(v))$ so that if M is a $\mathbb{Q}(v)$ module then $T_s^A = T_s$ on M .

We want to define the twisting functor for every $w \in W$ such that if w has a reduced expression $w = s_{i_r} \cdots s_{i_1}$ then $T_w^A = T_{s_{i_r}}^A \circ \cdots \circ T_{s_{i_1}}^A$. As before we define a 'semiregular bimodule' $S_A^w = U_A \otimes_{U_A^-(w)} U_A^-(w)^*$ and show this is a bimodule isomorphic to $S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$.

Theorem 4.5 $S_A^w := U_A \otimes_{U_A^-(w)} U_A^-(w)^*$ is a bimodule isomorphic to $S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$ and the functors T_s^A , $s \in \Pi$ satisfy braid relations.

Proof. Note that $U_A^-(w)$ can be seen as an A -submodule of $U_v^-(w)$ and similarly $U_A^-(w)^*$ can be seen as a submodule of $U_v^-(w)^*$. So we have an injective A homomorphism

$$S_A^w \rightarrow S_v^w.$$

Assume the length of w is r and $w = s_{i_r} w'$, $l(w') = r - 1$. We want to show that the isomorphism φ_r from Proposition 3.2 restricts to an isomorphism $S_A^w \rightarrow S_A(F_{\beta_r}) \otimes_{U_A} S_A^{w'}$.

Assume $f \in U_A^-(w)$ is such that $f = g \cdot f'_m$ meaning that $f(xF_{\beta_r}^{(n)}) = g(x)\delta_{m,n}$, ($x \in U_A^-(w')$, $n \in \mathbb{N}$) where $g \in U_A^-(w')^*$. Then $f'_m = [m]_{\beta_r}! f_m$ where f_m is defined like in Proposition 3.2 and for $u \in U_A$ we have therefore

$$\varphi_r(u \otimes f) = uF_{\beta_r}^{(-m)} F_{\beta_r}^{-1} \otimes (1 \otimes g)$$

which can be seen to lie in $S_A(F_{\beta_r}) \otimes_{U_A} S_A^{w'}$. The inverse also restricts to a map to the right space:

$$\begin{aligned} \psi_r(uF_{\beta_r}^{(-m)} F_{\beta_r}^{-1} \otimes (1 \otimes g)) &= \psi_r(u[m]_{\beta_r}! F_{\beta_r}^{-m-1} \otimes (1 \otimes g)) \\ &= [m]_{\beta_r}! u \otimes f_m \cdot g \\ &= u \otimes f'_m \cdot g. \end{aligned}$$

The maps are well defined because they are restrictions of well defined maps and it is easy to see that they are inverse to each other.

As in the generic case we get a right module action on S_A^w in this way. This is the right action coming from S_v^w restricted to S_A^w . So now we have $S_A^w = S_A(F_{\beta_r}) \otimes_{U_A} \cdots \otimes_{U_A} S_A(F_{\beta_1})$. Showing that the twisting functors then satisfy braid relations is done in the same way as in Proposition 3.11. \square

Now we can define $T_w^A = T_{s_{i_r}}^A \circ \cdots \circ T_{s_{i_1}}^A$ if $w = s_{i_r} \cdots s_{i_1}$ is a reduced expression of w . By the previous theorem there is no ambiguity in this definition since the T_s^A 's satisfy braid relations.

It is now possible for any A algebra R to define twisting functors $U_R\text{-Mod} \rightarrow U_R\text{-Mod}$. Just tensor over A with R .

F.x. let $R = \mathbb{C}$ with $v \mapsto 1$. $S_A(F_{\beta}) \otimes_A \mathbb{C}$ is just the normal $S^s = U_{(y_{\beta})}/U$ via the isomorphism $uF_{\beta}^{(-n)} F_{\beta}^{-1} \otimes 1 \mapsto \bar{u}y_{\beta}^{-n-1}$ where \bar{u} is given by the isomorphism between $U_A^- \otimes_A \mathbb{C}$ and U^- .

Theorem 4.6 Let R be an A -algebra with $v \in A$ being sent to $q \in R \setminus \{0\}$. Let $\lambda : U_R^0 \rightarrow R$ be an R -algebra homomorphism and let $M_R(\lambda) = U_R \otimes_{U_R^{\geq 0}} R_{\lambda}$ be

the U_R Verma module with highest weight λ where R_λ is the rank 1 free $U_R^{\geq 0}$ -module with $U_R^{>0}$ acting trivially and U_R^0 acting as λ . Let $D : U_R \rightarrow U_R$ be the duality functor on $U_R - \text{Mod}$ induced from the duality functor on $U_A \rightarrow U_A$. Then

$$T_{w_0}^R M_R(\lambda) \cong DM_R(w_0.\lambda).$$

Proof. The proof is the almost the same as the proof of Theorem 3.9. We have by Corollary 2.19 (setting $k = \langle \beta_j, \beta_r^\vee \rangle$)

$$F_{\beta_j} F_{\beta_r}^{(-a)} F_{\beta_r}^{-1} = q^{-(a+1)(\beta_r|\beta_j)} F_{\beta_r}^{(-a)} F_{\beta_r}^{-1} F_{\beta_j} + \sum_{j \geq 1} q_{\beta_r}^{-(a+1)k - (a+2)i} F_{\beta_r}^{(-a-i)} F_{\beta_r}^{-1} \widetilde{\text{ad}}(F_{\beta_r}^{(i)})(u).$$

Define for $M = (m_1, \dots, m_N) \in \mathbb{N}$ the function

$$g_M(F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) = \begin{cases} 1 & \text{if } a_1 = m_1 \dots a_N = m_N \\ 0 & \text{otherwise} \end{cases}.$$

Note that $g_{(0, \dots, 0)} = g_\lambda$ from Theorem 3.9. In particular it has weight λ . We want to show that $DM_R^{w_0}(\lambda) = U_R g_{(0, \dots, 0)}$. We use induction on the number of nonzero entries in M . Assume j is such that $m_N = \dots = m_{j+1} = 0$ and $m_j = n > 0$. Let $M' = (0, \dots, 0, m_{j-1}, \dots, m_1)$. By induction $g_{M'} \in U_R g_{(0, \dots, 0)}$.

Set $u = \omega(S^{-1}(R_{w_0}^{-1}(F_{\beta_j}^{(n)})))$. Then

$$\begin{aligned} & ug_{M'}(F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\ &= g_{M'}(F_{\beta_j}^{(n)} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) \\ &= g_{M'}\left(\frac{1}{[n]_{\beta_j}!} F_{\beta_j}^n F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}\right) \\ &= g_{M'}\left(q^{c_1} \frac{1}{[n]_{\beta_j}!} F_{\beta_j}^{n-1} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_j} F_{\beta_j}^{(-a_j)} F_{\beta_j}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}\right) \\ &\vdots \\ &= g_{M'}\left(q^{c_n} \frac{1}{[n]_{\beta_j}!} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes F_{\beta_j}^n F_{\beta_j}^{(-a_j)} F_{\beta_j}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}\right) \\ &= \begin{cases} g_{M'}(q^{c_n} F_{\beta_N}^{(-a_N)} F_{\beta_N}^{-1} \otimes \dots \otimes [a_j]_{\beta_j} F_{\beta_j}^{(-(a_j-n))} F_{\beta_j}^{-1} \otimes \dots \otimes F_{\beta_1}^{(-a_1)} F_{\beta_1}^{-1} \otimes v_{w_0.\lambda}) & \text{if } n \leq a_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for some appropriate integers $c_1, \dots, c_n \in \mathbb{Z}$. $g_{M'}$ is nonzero on this only when $n = a_j$. So we get in conclusion that $ug_{M'} = v^{-c_n} g_M$. This finishes the induction step. \square

5 \mathfrak{sl}_2 calculations

Assume $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Let $M_A(v^r)$ be the $U_A(\mathfrak{sl}_2)$ Verma module with highest weight $v^r \in \mathbb{Z}$ i.e. $M_A(v^r) = U_A \otimes_{U_A^{\geq 0}} A_{v^r}$ where A_{v^r} is the free $U_A^{\geq 0}$ -module of rank 1 with U_A^+ acting trivially and $K \cdot 1 = q^r$. Inspired by [And03] we see that in \mathfrak{sl}_2 we have for $r \in \mathbb{Z}$ the homomorphism $\varphi : M_A(v^r) \rightarrow DM_A(v^r)$ given by:

Let $\{w_i = F^{(i)}w_0\}$ be a basis for $M_A(\lambda)$ where w_0 is a highest weight vector in $M_A(v^r)$ and let $\{w_i^*\}$ be the dual basis in $DM_A(\lambda)$. Then

$$\varphi(w_i) = (-1)^i v^{i(i-1)-ir} \begin{bmatrix} r \\ i \end{bmatrix} w_i^*.$$

Checking that this is indeed a homomorphism of U_A algebras is a straightforward calculation.

By Theorem 4.6 we see that $DM_A(v^r) = M_A^s(v^r)$. In the following section we will try to say something about the composition factors of a Verma module so it is natural to consider first \mathfrak{sl}_2 Verma modules.

Definition 5.1 Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Then $H_A(v^r)$ is defined to be the free $U_A(\mathfrak{sl}_2)$ -module of rank $r+1$ with basis e_0, \dots, e_r defined as follows:

$$\begin{aligned} Ke_i &= v^{r-2i} e_i, \quad \begin{bmatrix} K; c \\ t \end{bmatrix} e_i = \begin{bmatrix} r-2i+c \\ t \end{bmatrix} e_i \\ E^{(n)} e_i &= \begin{bmatrix} i \\ n \end{bmatrix} e_{i-n}, \quad n \in \mathbb{N} \\ F^{(n)} e_i &= \begin{bmatrix} r-i \\ n \end{bmatrix} e_{i+n}, \quad n \in \mathbb{N} \end{aligned}$$

for $i = 0, \dots, r$. Where $e_{<0} = 0 = e_{>r}$.

Lemma 5.2 Let $\mathfrak{g} = \mathfrak{sl}_2$. Let $r \in \mathbb{N}$. Then we have a short exact sequence:

$$0 \rightarrow DM_A(v^{-r-2}) \rightarrow M_A(v^r) \rightarrow H_A(v^r) \rightarrow 0.$$

Proof. We use the fact that $DM_A(v^{-r-2}) = T_s^A M_A(v^r)$ by Theorem 4.6. Let $e_i = F^{(i)}w_0$ where w_0 is a highest weight vector in $M_A(v^r)$. We will construct a U_A -homomorphism $\text{span}_A \{e_i | i > r\} \rightarrow DM_A(-r-2)$. Let τ be as defined in [Jan96] Chapter 4. Note that in $U_{A(F)}$ $S(\tau(F))$ is invertible so we can consider S and τ as automorphisms of $U_{A(F)}$. We define a map by

$$e_{r+i} \mapsto (-1)^{r+i} S(\tau(F^{(-i-1)}))w_0$$

Note that for \mathfrak{sl}_2 $R_s = S \circ \tau \circ \omega$. Using this and the formula in Proposition 4.2 it is straightforward to check that this is a U_A -homomorphism. \square

If we specialize to an A -algebra R with R being a field where v is sent to a non-root of unity $q \in R$ we get that $M_R(q^k) = U_R \otimes_{U_A} M_A(v^k)$ is simple for $k < 0$. So in the above with $r \in \mathbb{N}$, $DM_R(q^{-r-2}) = M_R(q^{-r-2}) = L_R(q^{-r-2})$ and actually we see also that $H_R(q^r) = L_R(q^r)$. So there is an exact sequence

$$0 \rightarrow L_R(q^{-r-2}) \rightarrow M_R(q^r) \rightarrow L_R(q^r) \rightarrow 0.$$

So the composition factors in $M_R(q^r)$ are $L_R(q^r)$ and $L_R(q^{-r-2}) = L_R(s.q^r)$ where s is the simple reflection in the Weyl group of \mathfrak{sl}_2 .

6 Jantzen filtration

In this section we will work with the field \mathbb{C} and send v to a non root of unity $q \in \mathbb{C}^*$. We define $U_q = U_A \otimes_A \mathbb{C}_q$ where \mathbb{C}_q is the A -algebra \mathbb{C} with v being sent to q . These results compare to the results in [And03] and [AL03].

Let λ be a weight i.e. an algebra homomorphism $U_q^0 \rightarrow \mathbb{C}$ and let $M(\lambda) = U_q \otimes_{U_q^0} \mathbb{C}_\lambda$ be the Verma module of highest weight λ . Consider the local ring $B = \mathbb{C}[X]_{(X-1)}$ and the quantum group $U_B = U_A \otimes_A B$. We define $\lambda X : U_q^0 \rightarrow B$ to be the weight defined by $(\lambda X)(K_\mu) = \lambda(K_\mu)X$ and we define $M_B(\lambda X) = U_B \otimes_{U_B^0} B_{\lambda X}$ to be the Verma module with highest weight λX . Note that $M_B(\lambda X) \otimes_B \mathbb{C} \cong M(\lambda)$ when we consider \mathbb{C} as a B -algebra via the specialization $X \mapsto 1$.

For a simple root $\alpha_i \in \Pi$ we define $M_{B,i}(\lambda X) := U_B(i) \otimes_{U_B^0} B_{\lambda X}$, where $U_B(i)$ is the subalgebra generated by $U_B^{\geq 0}$ and F_{α_i} . We define $M_{B,i}^{s_i}(\lambda) := {}^{s_i}((U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B(i)} M_{B,i}(s_i \cdot \lambda))$ where the module $(U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*)$ is a $U_B(i)$ -bimodule isomorphic to $S_{B,i}(F_{\alpha_i}) = (U_B(i))_{(F_{\alpha_i})}/U_B(i)$ by similar arguments as earlier.

Proposition 6.1 *There exists a nonzero homomorphism $\varphi : M_B(\lambda X) \rightarrow M_B^{s_\alpha}(\lambda X)$ which is an isomorphism if $q^\rho \lambda(K_\alpha) \notin \pm q_{\alpha}^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence*

$$0 \rightarrow M_B(\lambda X) \xrightarrow{\varphi} M_B^{s_\alpha}(\lambda X) \rightarrow M(s_\alpha \cdot \lambda) \rightarrow 0$$

where we have identified the cokernel $M_B^{s_\alpha}(s_\alpha \cdot \lambda X)/(X-1)M_B(s_\alpha \cdot \lambda X)$ with $M(s_\alpha \cdot \lambda)$.

Furthermore there exists a nonzero homomorphism $\psi : M_B^{s_\alpha}(\lambda X) \rightarrow M_B(\lambda X)$ which is an isomorphism if $q^\rho \lambda(K_\alpha) \notin \pm q_{\alpha}^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence

$$0 \rightarrow M_B^{s_\alpha}(\lambda X) \xrightarrow{\psi} M_B(\lambda X) \rightarrow M(\lambda)/M(s_\alpha \cdot \lambda) \rightarrow 0.$$

Proof. We will first define a map from $M_{B,i}(\lambda X)$ to

$$M_{B,i}^{s_i}(\lambda X) = {}^{s_i}((U_B(i))_{(F_\alpha)}/U_B(i) \otimes_{U_B} M_{B,i}(s_\alpha \cdot \lambda X)).$$

Setting $\lambda' = \lambda X$ define

$$\varphi(F_\alpha^{(n)} v_{\lambda'}) = a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'}$$

where

$$a_n = (-1)^n q_\alpha^{-n(n+1)} \lambda'(K_\alpha)^n \prod_{t=1}^n \frac{q_\alpha^{1-t} \lambda'(K_\alpha) - q_\alpha^{t-1} \lambda'(K_\alpha)^{-1}}{q_\alpha^t - q_\alpha^{-t}}.$$

So we need to check that this is a homomorphism: First of all for $\mu \in Q$.

$$\begin{aligned}
K_\mu \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} &= a_n K_{s_\alpha(\mu)} F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= q^{(n+1)(s_\alpha(\mu)|\alpha)} (s_\alpha \cdot \lambda') (K_{s_\alpha(\mu)}) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= q^{-(n+1)(\mu|\alpha)} q^{-(\rho|s_\alpha(\mu))} q^{(\rho|\mu)} \lambda' (K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= q^{-(n+1)(\mu|\alpha)} q^{-(\rho-\alpha|\mu)} q^{(\rho|\mu)} \lambda' (K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= q^{-n(\mu|\alpha)} \lambda' (K_\mu) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= \varphi(K_\mu F_\alpha^{(n)} v_{\lambda'}).
\end{aligned}$$

We have

$$\begin{aligned}
E_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} &= a_n R_{s_i}(E_{\alpha_i}) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -a_n F_\alpha K_\alpha F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -q_\alpha^{2(n+1)} s_\alpha \cdot \lambda' (K_\alpha) [n]_\alpha a_n F_\alpha^{(-n+1)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -q_\alpha^{2n} \lambda' (K_\alpha^{-1}) [n]_\alpha a_n F_\alpha^{(-n+1)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(E_\alpha F_\alpha^{(n)} v_{\lambda'}) &= \varphi \left(F_\alpha^{(n-1)} \frac{q_\alpha^{1-n} K_\alpha - q_\alpha^{n-1} K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} v_{\lambda'} \right) \\
&= \left(a_{n-1} F_\alpha^{(-n+1)} F_\alpha^{-1} \frac{q_\alpha^{1-n} \lambda' (K_\alpha) - q_\alpha^{n-1} \lambda' (K_\alpha)^{-1}}{q_\alpha - q_\alpha^{-1}} \right) \otimes v_{\lambda'}
\end{aligned}$$

so we see that $\varphi(E_\alpha F_\alpha^{(n)} v_{\lambda'}) = E_\alpha \cdot \varphi(F_\alpha^{(n)} v_{\lambda'})$. Clearly $\varphi(E_{\alpha'} F_\alpha^{(n)} v_{\lambda'}) = 0 = E_{\alpha'} \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{\lambda'}$ for any simple $\alpha' \neq \alpha$ so what we have left is F_α : By Proposition 4.3

$$\begin{aligned}
F_\alpha \cdot a_n F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} &= a_n R_{s_i}(F_\alpha) F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -a_n K_\alpha^{-1} E_\alpha F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -a_n K_\alpha^{-1} F_\alpha^{(-n-1)} F_\alpha^{-1} [K_\alpha; n+2] \otimes v_{s_\alpha \cdot \lambda'} \\
&= -a_n q_\alpha^{-2(n+2)} s_\alpha \cdot \lambda' (K_\alpha^{-1}) \frac{q_\alpha^{n+2} s_\alpha \cdot \lambda' (K_\alpha) - q_\alpha^{-n-2} s_\alpha \cdot \lambda' (K_\alpha)^{-1}}{q_\alpha - q_\alpha^{-1}} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} \\
&= -a_n q_\alpha^{-2(n+1)} \lambda' (K_\alpha) \frac{q_\alpha^n \lambda' (K_\alpha^{-1}) - q_\alpha^{-n} \lambda' (K_\alpha)}{q_\alpha - q_\alpha^{-1}} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) &= [n+1]_\alpha \varphi(F_\alpha^{(n+1)} v_\lambda) \\
&= [n+1]_\alpha a_{n+1} F_\alpha^{(-n-1)} F_\alpha^{-1} \otimes v_\lambda
\end{aligned}$$

so we see that $\varphi(F_\alpha F_\alpha^{(n)} v_\lambda) = F_\alpha \cdot \varphi(F_\alpha^{(n)} v_\lambda)$.

Now note that if $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ then $X-1$ does not divide a_n for any $n \in \mathbb{N}$ implying that a_n is a unit. So when $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$, φ is an isomorphism. If

$\lambda(K_\alpha) = \varepsilon q_\alpha^r$ for some $\varepsilon \in \{\pm 1\}$ and $r \in \mathbb{N}$ we see that $X - 1$ divides a_n for any $n > r$ so the image of φ is

$$\text{span}_B \left\{ F_\alpha^{(-n)} F_\alpha \otimes v_{s_\alpha \cdot \lambda'} | n \leq r \right\} + (X - 1) \text{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} | n > r \right\}.$$

Thus the cokernel $M_{B,i}^{s_i}(\lambda) / \text{Im } \varphi$ is equal to

$$\text{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} | n > r \right\} / (X - 1) \text{span}_B \left\{ F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'} | n > r \right\}$$

which is seen to be isomorphic to $M_{B,i}^{s_i}(s_i \cdot \lambda') / (X - 1) M_{B,i}^{s_i}(s_i \cdot \lambda')$.

If $\lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{N}}$ then obviously we can define an inverse to φ , $\psi : M_{B,i}^{s_i}(\lambda') \rightarrow M_{B,i}(\lambda')$. If $\lambda(K_\alpha) = \varepsilon q^r$ for some $\varepsilon \in \{\pm 1\}$ and some $r \in \mathbb{N}$ we define $\psi : M_{B,i}^{s_i}(\lambda') \rightarrow M_{B,i}(\lambda')$ by

$$\psi(F_\alpha^{(-n)} F_\alpha^{-1} \otimes v_{s_\alpha \cdot \lambda'}) = \frac{(X - 1)}{a_n} F_\alpha^{(n)} v_{\lambda'}$$

(note that for all λ and all $n \in \mathbb{N}$, $(X - 1)^2 \nmid a_n$ so $\frac{(X-1)}{a_n} \in B$). This implies $\varphi \circ \psi = (X - 1) \text{id}$ and $\psi \circ \varphi = (X - 1) \text{id}$. Using that φ is a U_q -homomorphism we show that ψ is: For $u \in U_q$ and $v \in M_{B,i}^{s_i}(\lambda')$:

$$(X - 1)\psi(uv) = \psi(u\varphi(\psi(v))) = \psi(\varphi(u\psi(v))) = (X - 1)u\psi(v).$$

Since B is a domain this implies $\psi(uv) = u\psi(v)$.

We see that $X - 1$ divides $\frac{X-1}{a_n}$ for any $n \leq r$ so the image of ψ is

$$(X - 1) \text{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n \leq r \right\} + \text{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n > r \right\}.$$

Thus the cokernel $M_{B,i}(\lambda) / \text{Im } \psi$ is equal to

$$\text{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n \leq r \right\} / (X - 1) \text{span}_B \left\{ F_\alpha^{(n)} v_{\lambda'} | n \leq r \right\}$$

which is seen to be isomorphic to

$$M_{B,i}(\lambda) / M_{B,i}(s_\alpha \cdot \lambda).$$

Now we induce to the whole quantum group: We have that

$$M_B(\lambda') = U_B \otimes_{U_B(i)} M_{B,i}(\lambda')$$

and

$$\begin{aligned} M_B^{s_i}(\lambda') &= {}^{s_i} \left((U_B \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B} U_B \otimes_{U_B^{\geq 0}} B_{\lambda'} \right) \\ &\cong {}^{s_i} \left((U_B \otimes_{U_B(i)} U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B^{\geq 0}} B_{\lambda'} \right) \\ &\cong U_B \otimes_{U_B(i)} {}^{s_i} \left((U_B(i) \otimes_{U_B(s_i)} U_B(s_i)^*) \otimes_{U_B^{\geq 0}} B_{\lambda'} \right) \\ &\cong U_B \otimes_{U_B(i)} M_{B,i}^{s_i}(\lambda') \end{aligned}$$

so by inducing to U_B -modules using the functor $U_B \otimes_{U_B(i)} -$ we get a map $\varphi : M_B(\lambda') \rightarrow M_B^{s_i}(\lambda')$ and a map $\psi : M_B^{s_i}(\lambda') \rightarrow M_B(\lambda')$. This functor is exact on $M_{B,i}(\lambda')$ and $M_{B,i}^{s_i}(\lambda')$ so the proposition follows from the above calculations. \square

Proposition 6.2 *Let $\lambda : U_q^0 \rightarrow \mathbb{C}$ be a weight. Set $\lambda' = \lambda X$. Let $w \in W$ and $\alpha \in \Pi$ such that $w(\alpha) > 0$. There exists a nonzero homomorphism $\varphi : M_B^w(\lambda') \rightarrow M_B^{ws_\alpha}(\lambda')$ that is an isomorphism if $q^\rho \lambda(K_{w(\alpha)}) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have the short exact sequence*

$$0 \rightarrow M_B^w(\lambda') \xrightarrow{\varphi} M_B^{ws_\alpha}(\lambda') \rightarrow M^w(s_{w(\alpha)}.\lambda) \rightarrow 0$$

where the cokernel $M_B^{ws_\alpha}(s_{w(\alpha)}.\lambda')/(X-1)M_B^{ws_\alpha}(s_{w(\alpha)}.\lambda')$ is identified with $M^w(s_{w(\alpha)}.\lambda)$.

Furthermore there exists a nonzero homomorphism $\psi : M_B^{ws_\alpha}(\lambda X) \rightarrow M_B^w(\lambda X)$ which is an isomorphism if $q^\rho \lambda(K_{w(\alpha)}) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have a short exact sequence

$$0 \rightarrow M_B^{ws_\alpha}(\lambda') \xrightarrow{\psi} M_B^w(\lambda') \rightarrow M^w(\lambda)/M^w(s_{w(\alpha)}.\lambda) \rightarrow 0.$$

Proof. Let $\mu = w^{-1}.\lambda$ and $\mu' = \mu X$ then from Proposition 6.1 we get a homomorphism $M_B(\mu') \rightarrow M_B^{s_\alpha}(\mu')$ and a homomorphism $M_B^{s_\alpha}(\mu') \rightarrow M_B(\mu')$. Observe that

$$\begin{aligned} q^\rho \mu(K_\alpha) &= w^{-1}.\lambda(K_\alpha) \\ &= w^{-1}(q^\rho \lambda)(K_\alpha) \\ &= q^{(\rho|w(\alpha))} \lambda(K_{w(\alpha)}) \\ &= (q^\rho \lambda)(K_{w(\alpha)}) \end{aligned}$$

so $M_B(\mu') \rightarrow M_B^{s_\alpha}(\mu')$ and $M_B^{s_\alpha}(\mu') \rightarrow M_B(\mu')$ are isomorphisms if $(q^\rho \lambda)(K_{w(\alpha)}) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have the short exact sequences

$$0 \rightarrow M_B(\mu') \rightarrow M_B^{s_\alpha}(\mu') \rightarrow M(\mu') \rightarrow 0$$

and

$$0 \rightarrow M_B^{s_\alpha}(\mu') \rightarrow M_B(\mu') \rightarrow M(\mu')/M(s_\alpha.\mu') \rightarrow 0.$$

Now we use the twisting functor T_w on the homomorphisms $M_B(\mu') \rightarrow M_B^{s_\alpha}(\mu')$ and $M_B^{s_\alpha}(\mu') \rightarrow M_B(\mu')$ to get homomorphisms $\varphi : M_B^w(\lambda) \rightarrow M_B^{ws_\alpha}(\lambda)$ and $\psi : M_B^{ws_\alpha}(\lambda) \rightarrow M_B^w(\lambda)$ (using the fact that $T_w \circ T_{s_\alpha} = T_{ws_\alpha}$). We are done if we show that T_w is exact on Verma modules. But

$$\begin{aligned} T_w M_B(\mu') &= {}^w \left((U_B^-(w)^* \otimes_{U_B^-(w)} U_B) \otimes_{U_B} U_B \otimes_{U_B^{\geq 0}} B_{\mu'} \right) \\ &\cong {}^w \left((U_B^-(w)^* \otimes_{U_B^-(w)} U_B) \otimes_{U_B^{\geq 0}} B_{\mu'} \right) \\ &\cong {}^w \left(U_B^-(w)^* \otimes_{U_B^-(w)} U_B^- \otimes_{\mathbb{C}} B_{\mu'} \right) \end{aligned}$$

as vectorspaces and U_B^0 modules. Observing that U_B^- is free over $U_B^-(w)$ we get the exactness. \square

Fix a weight $\lambda : U_q^0 \rightarrow \mathbb{C}$ and a $w \in W$. Define $\Phi^+(w) := \Phi^+ \cap w(\Phi^-) = \{\beta \in \Phi^+ | w^{-1}(\beta) < 0\}$ and $\Phi^+(\lambda) := \{\beta \in \Phi^+ | q^\rho \lambda(K_\beta) \in \pm q_\beta^{\mathbb{Z}}\}$. Choose a

reduced expression of $w_0 = s_{i_1} \cdots s_{i_N}$ such that $w = s_{i_n} \cdots s_{i_1}$. Set

$$\beta_j = \begin{cases} -ws_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j \leq n \\ ws_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j > n. \end{cases}$$

Then $\Phi^+ = \{\beta_1, \dots, \beta_N\}$ and $\Phi^+(w) = \{\beta_1, \dots, \beta_n\}$. We denote by $\Psi_B^w(\lambda)$ the composite

$$M_B^w(\lambda X) \xrightarrow{\varphi_1^w(\lambda)} M_B^{ws_{i_1}}(\lambda X) \xrightarrow{\varphi_2^w(\lambda)} \cdots \xrightarrow{\varphi_N^w(\lambda)} M_B^{ww_0}(\lambda X)$$

where the homomorphisms are the ones from Proposition 6.2 i.e. the first n homomorphisms are the ψ 's and the last $N - n$ homomorphisms are the φ 's from Proposition 6.2. We denote by $\Psi^w(\lambda)$ the U_q -homomorphism $M^w(\lambda X) \rightarrow M^{ww_0}(\lambda X)$ induced by tensoring the above U_B -homomorphism with \mathbb{C} considered as a B module by $X \mapsto 1$.

In analogy with Theorem 7.1 in [AL03] and Proposition 4.1 in [And03] we have

Theorem 6.3 *Let $\lambda : U_q^0 \rightarrow \mathbb{C}$ be a weight. Let $w \in W$. Then there exists a filtration of $M^w(\lambda)$, $M^w(\lambda) \supset M^w(\lambda)^1 \supset \cdots \supset M^w(\lambda)^r$ such that $M^w(\lambda)/M^w(\lambda)^1 \cong \text{Im } \Psi^w(\lambda) \subset M^{ww_0}(\lambda)$ and*

$$\begin{aligned} \sum_{i=1}^r \text{ch } M^w(\lambda)^i &= \sum_{\beta \in \Phi^+(\lambda) \cap \Phi^+(w)} (\text{ch } M(\lambda) - \text{ch } M(s_\beta \lambda)) \\ &+ \sum_{\beta \in \Phi^+(\lambda) \setminus \Phi^+(w)} \text{ch } M(s_\beta \lambda). \end{aligned}$$

Proof. Set $\lambda' = \lambda X$. Define for $i \in \mathbb{N}$

$$M_B^w(\lambda')^i = \{m \in M_B^w(\lambda') \mid \Psi_B^w(\lambda)(m) \in (X - 1)^i M_B^{ww_0}(\lambda')\}.$$

Set $M^w(\lambda)^i = \pi(M_B^w(\lambda')^i)$ where $\pi : M_B^w(\lambda) \rightarrow M^w(\lambda)$ is the canonical homomorphism from $M_B^w(\lambda)$ to $M_B^w(\lambda)/(X - 1)M_B^w(\lambda) \cong M^w(\lambda)$. This defines a filtration of $M^w(\lambda)$. We have $M^w(\lambda)^{N+1} = 0$ so the filtration is finite.

Let $\mu : U_q^0 \rightarrow \mathbb{C}$ be a weight. Set $\mu' = \mu X$. The maps $\varphi_j^w(\lambda)$ restrict to weight spaces. Denote the restriction $\varphi_j^w(\lambda)_{\mu'}$. Let $\Psi_B^w(\lambda)_{\mu'} : M_B^w(\lambda)_{\mu'} \rightarrow M_B^{ww_0}(\lambda)_{\mu'}$ be the restriction of $\Psi_B^w(\lambda)$ to the μ' weight space. We have a nondegenerate bilinear form $(-, -)$ on $M(\lambda')_{\mu'}$ given by $(x, y) = (\Psi_B^w(\lambda)_{\mu'}(x))(y)$. It is nondegenerate since $\Psi_B^w(\lambda)$ is injective. Let $\nu : B \rightarrow \mathbb{C}$ be the $(X - 1)$ -adic valuation i.e. $\nu(b) = m$ if $b = (X - 1)^m b'$, $(X - 1) \nmid b'$. We have by [Hum08, Lemma 5.6] (originally Lemma 5.1 in [Jan79])

$$\sum_{j \geq 1} \dim(M_j)_\mu = \nu(\det \Psi_B^w(\lambda)_{\mu'}).$$

Clearly $\nu(\det \Psi_B^w(\lambda)_{\mu'}) = \sum_{j=1}^N \nu(\det \varphi_j^w(\lambda)_{\mu'})$ and the result follows when we show:

$$\nu(\det \varphi_j^w(\lambda)_{\mu'}) = \dim_{\mathbb{C}} (\text{coker } \varphi_j^w(\lambda)_{\mu'}).$$

Fix $\varphi := \varphi_j^w(\lambda)_{\mu'}$ and let M and N be the domain and codomain respectively. M and N are free B modules of finite rank. Let d be the rank. We can choose

bases m_1, \dots, m_d and n_1, \dots, n_d such that $\varphi(m_i) = a_i n_i$, $i = 1, \dots, d$ for some $a_i \in B$. Set $C = \text{coker } \varphi \cong \bigoplus_{i=1}^d B/(a_i)$ and set $C_{\mathbb{C}} = C \otimes_B (B/(X-1)B) = C \otimes_B \mathbb{C}$ where \mathbb{C} is considered a B -module by $X \mapsto 1$. Note that

$$B/(a_i) \otimes_B \mathbb{C} = \begin{cases} \mathbb{C}, & \text{if } (X-1)|a_i \\ 0, & \text{otherwise} \end{cases}$$

so $\dim_{\mathbb{C}} C_{\mathbb{C}} = \#\{i | \nu(a_i) > 0\}$. Since there exists a $\psi : N \rightarrow M$ such that $\varphi \circ \psi = (X-1)\text{id}$ we get $\nu(a_i) \leq 1$ for all i . So then $\dim_{\mathbb{C}} C_{\mathbb{C}} = \nu(\det \varphi)$ and the claim has been shown. \square

7 Linkage principle

Let R be a field that is an A -algebra and $q \in R$ the nonzero element that v is sent to. As usual we can define the Verma modules: Assume $\lambda : U_R^0 \rightarrow R$ is a homomorphism. Then we define $M_R(\lambda) = U_R \otimes_{U_R^{\geq 0}} R_{\lambda}$ where R_{λ} is the onedimensional R -module with trivial action from U_R^+ and U_R^0 acting as λ . There is a unique simple quotient $L_R(\lambda)$ of $M_R(\lambda)$.

Let $\alpha = \alpha_i \in \Pi$. Consider the parabolic Verma module $M_{R,i}(\lambda) := U_R(i) \otimes_{U_R^{\geq 0}} R_{\lambda}$, where $U_R(i)$ is the submodule generated by $U_R^{\geq 0}$ and F_{α_i} . We get a map $M_{R,i}(\lambda) \rightarrow M_{R,i}^s(\lambda) := {}^s((U_R(i) \otimes_{U_R(s_i)} U_R(s_i)^*) \otimes_{U_R(i)} M_{R,i}(s.\lambda))$ where the module $(U_R(i) \otimes_{U_R(s_i)} U_R(s_i)^*)$ is a $U_R(i)$ -bimodule by the similar arguments as earlier. Inducing to the whole quantum group and using T_w we get a homomorphism

$$M_R^w(\lambda) \rightarrow M_R^{ws_{\alpha}}(\lambda)$$

So we can construct a sequence of homomorphisms $\varphi_1, \dots, \varphi_N$

$$M_R(\lambda) \xrightarrow{\varphi_1} M_R^{s_{i_1}}(\lambda) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_N} M_R^{w_0}(\lambda) = DM_R(\lambda).$$

We denote the composition by Ψ . Note that the image of Ψ must be the unique simple quotient $L_R(\lambda)$ of $M_R(\lambda)$ since every map $M(\lambda) \rightarrow DM(\lambda)$ maps to the unique simple quotient of $M(\lambda)$ (by the usual arguments e.g. like in [Hum08, Theorem 3.3]).

First we want to consider some facts about the map $\varphi : M_R^w(\lambda) \rightarrow M_R^{ws_{\alpha}}(\lambda)$. Let $M_{\alpha}(\lambda)$ denote the $U_R(\mathfrak{sl}(2))$ Verma module with highest weight $\lambda(K_{\alpha})$. We will use the notation $M_{p_{\alpha}}(\lambda)$ for the parabolic $U_R(i)$ Verma module $U_R(i) \otimes_{U_R^{\geq 0}} R_{\lambda}$. The map φ was constructed by first inducing the map of parabolic modules and then using the twisting functor T_w .

Assume the sequence of $U_R(\mathfrak{sl}_2)$ modules $M_{\alpha}(\lambda) \rightarrow M_{\alpha}^s(\lambda) \rightarrow Q_{\alpha}(\lambda) \rightarrow 0$ is exact (i.e. $Q_{\alpha}(\lambda)$ is the cokernel of the map $M_{\alpha}(\lambda) \rightarrow M_{\alpha}^s(\lambda)$). Inflating to the parabolic situation we get an exact sequence $M_{p_{\alpha}}(\lambda) \rightarrow M_{p_{\alpha}}^s(\lambda) \rightarrow Q_{p_{\alpha}}(\lambda) \rightarrow 0$ where $Q_{p_{\alpha}}(\lambda)$ is just the inflation of $Q_{\alpha}(\lambda)$ to the corresponding parabolic module.

Inducing from a parabolic module to the whole module is done by applying the functor $M \mapsto U_R \otimes_{U(i)} M$. This is right exact so we get the exact sequence $M_R(\lambda) \rightarrow M_R^s(\lambda) \rightarrow Q_R(\lambda) \rightarrow 0$ where $Q_R(\lambda) = U_R \otimes_{U_R(i)} Q_{p_{\alpha}}(\lambda)$.

Assume we have a finite filtration of $Q_\alpha(\lambda)$:

$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q_\alpha(\lambda)$$

such that $Q_{i+1}/Q_i \cong L_\alpha(\mu_i)$. So we have after inflating:

$$0 = Q_{p_\alpha,0} \subset Q_{p_\alpha,1} \subset \cdots \subset Q_{p_\alpha,r} = Q_{p_\alpha}(\lambda)$$

such that $Q_{p_\alpha,i+1}/Q_{p_\alpha,i} \cong L_{p_\alpha}(\mu_i)$.

That is we have short exact sequences of the form

$$0 \rightarrow Q_{p_\alpha,i} \rightarrow Q_{p_\alpha,i+1} \rightarrow L_{p_\alpha}(\mu_i) \rightarrow 0.$$

Since induction is right exact we get the exact sequence

$$Q_{R,i} \rightarrow Q_{R,i+1} \rightarrow \overline{L_{p_\alpha}(\mu_i)} \rightarrow 0$$

where $Q_{R,i}$ is the induced module of $Q_{p_\alpha,i}$ and $\overline{L_{p_\alpha}(\mu_i)}$ is the induced module of $L_{p_\alpha}(\mu_i)$.

Starting from the top we have

$$Q_{R,r-1} \rightarrow Q_R(\lambda) \rightarrow \overline{L_{p_\alpha}(\mu_{r-1})} \rightarrow 0$$

so we see that the composition factors of $Q_R^{s_\alpha}(\lambda)$ are contained in the set of composition factors of $\overline{L_{p_\alpha}(\mu_{r-1})}$ and the composition factors of $Q_{R,r-1}$. By induction we get then that the composition factors of $Q_{R,r-1}$ are composition factors of $\overline{L_{p_\alpha}(\mu_i)}$, $i = 0, \dots, r-2$. The conclusion is that we can get a restriction on the composition factors of $Q_R(\lambda)$ by examining the composition factors of induced simple modules.

Let $L = L_{p_\alpha}(\mu)$ be a simple parabolic module and let \overline{L} be the induction of L . Then because induction is right exact we have

$$M_R(\mu) \rightarrow \overline{L} \rightarrow 0.$$

So the composition factors of \overline{L} are composition factors of $M_R(\mu)$. This gives us a restriction on the composition factors of $M_R(\lambda)$:

Use the above with $w^{-1}.\lambda$ in place of λ and use the twisting functor T_w^R on the exact sequence $M_R(w^{-1}.\lambda) \rightarrow M_R^s(w^{-1}.\lambda) \rightarrow Q_R(w^{-1}.\lambda) \rightarrow 0$ to get

$$M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda) \rightarrow Q_R^w(\lambda) \rightarrow 0$$

where $Q_R^{ws}(\lambda) = T_w^R(Q_R(w^{-1}.\lambda))$. Add the kernel to get the 4-term exact sequence

$$0 \rightarrow K_R^{ws}(\lambda) \rightarrow M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda) \rightarrow Q_R^{ws}(\lambda) \rightarrow 0$$

Since $\text{ch } M_R^w(\lambda) = \text{ch } M_R^{ws}(\lambda)$ we must have $\text{ch } K_R^{ws}(\lambda) = \text{ch } Q_R^{ws}(\lambda)$.

So we have a sequence of homomorphisms φ_i

$$M_R(\lambda) \xrightarrow{\varphi_1} M_R^s(\lambda) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_N} M_R^{w_0}(\lambda) = DM_R(\lambda)$$

and these maps each fit into a 4-term exact sequence

$$0 \rightarrow K_R^{ws}(\lambda) \rightarrow M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda) \rightarrow Q_R^{ws}(\lambda) \rightarrow 0$$

where $\text{ch } K_R^{ws}(\lambda) = \text{ch } Q_R^{ws}(\lambda)$. In particular $M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda)$ is an isomorphism if the corresponding \mathfrak{sl}_2 map $M_\alpha(w^{-1}.\lambda) \rightarrow DM_\alpha(w^{-1}.\lambda)(= M_\alpha^s(w^{-1}.\lambda))$ is an isomorphism. If the \mathfrak{sl}_2 map is not an isomorphism then we have a restriction on the composition factors that can get killed by the map $M_R(w^{-1}.\lambda) \rightarrow M_R^s(w^{-1}.\lambda)$ by the above. To get to the map $M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda)$ we use T_w which is right exact so we get a restriction on the composition factors killed by $M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda)$ too:

Fix α . From the above we know that a composition factor of $Q_R(\lambda)$ is a composition factor of $\overline{L_{p_\alpha}(\mu)}$ for some μ where $L_\alpha(\mu)$ is a composition factor of $M_\alpha(\lambda)$. Use this for $w^{-1}.\lambda$ and use T_w . So we get that a composition factor of $Q_R^{ws}(\lambda)$ is a composition factor of $T_w \overline{L_{p_\alpha}(\mu)}$ with μ as before. Since T_w is right exact we have that

$$T_w M_R(\mu) \rightarrow T_w \overline{L_{p_\alpha}(\mu)} \rightarrow 0$$

is exact. Since $\text{ch } T_w M_R(\mu) = \text{ch } M_R(w.\mu)$ we see that a composition factor of $Q_R^{ws}(\lambda)$ must be a composition factor of a Verma module $M_R(w.\mu)$ where μ is such that $L_\alpha(\mu)$ is a composition factor of $M_\alpha(w^{-1}.\lambda)$.

Definition 7.1 We define a partial order on weights. We say $\mu \leq \lambda$ if $\mu^{-1}\lambda = q^{\sum_{i=1}^n a_i \alpha_i}$ for some $a_i \in \mathbb{N}$ where $\mu^{-1} : U_R^0 \rightarrow \mathbb{C}$ is the weight with $\mu^{-1}(K_\alpha) = \mu(K_\alpha^{-1})$ for all $\alpha \in \Pi$.

For a weight ν of the form $\nu = q^{\sum_{i=1}^n a_i \alpha_i}$ with $a_i \in \mathbb{N}$ we call $\sum_{i=1}^n a_i$ the height of ν .

Note that for a Verma module $M(\lambda)$ we have $\mu \leq \lambda$ for all $\mu \in \text{wt } M(\lambda)$ where $\text{wt } M(\lambda)$ denotes the weights of $M(\lambda)$.

Definition 7.2 Let $\mu, \lambda \in \Lambda$. Define $\mu \uparrow_R \lambda$ to be the partial order induced by the following: μ is less than λ if there exists a $w \in W$, $\alpha \in \Pi$ and $\nu \in \Lambda$ such that $\mu = w.\nu < \lambda$ and $L_\alpha(\nu)$ is a composition factor of $M_\alpha(w^{-1}.\lambda)$.

i.e. $\mu \uparrow_R \lambda$ if there exists a sequence of weights $\mu = \mu_1, \dots, \mu_r = \lambda$ such that μ_i is related to μ_{i+1} as above.

We have established the following:

Proposition 7.3 If $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ then $\mu \uparrow_R \lambda$.

Proof. Choose a reduced expression of w_0 and construct the maps φ_i as above. If $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ it must be killed by one of the maps φ_i since the image of Ψ is $L_R(\lambda)$. So $L_R(\mu)$ must be a composition factor of one of the modules $Q_R^w(\lambda)$. We make an induction on the height of $\mu^{-1}\lambda$. If $\mu^{-1}\lambda = 1$ then $\lambda = \mu$ and we are done. Otherwise we see that $L_R(\mu)$ is a composition factor of one of the $Q_R^w(\lambda)$'s. But every composition factor of $Q_R^w(\lambda)$ is a composition factor of $M(\nu)$ where $\nu \uparrow_R \lambda$ and $\nu < \lambda$. Since $\nu < \lambda$ the height of $\mu^{-1}\nu$ is less than the height of $\mu^{-1}\lambda$ so we are done by induction. \square

In the non-root of unity case \uparrow_R is equivalent to the usual strong linkage: μ is strongly linked to λ if there exists a sequence μ_i with $\mu = \mu_1 < \mu_2 < \dots <$

$\mu_r = \lambda$ and $\mu_i = s_{\beta_i} \cdot \mu_{i+1}$ for some positive roots β_i (remember that if $\beta = w(\alpha)$ then $s_\beta = ws_\alpha w^{-1}$).

In the nonroot of unity case we see that $M_\alpha(w^{-1} \cdot \lambda)$ is simple if

$$q^\rho w^{-1} \cdot \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}.$$

Otherwise there is one composition factor in $M_\alpha(w^{-1} \cdot \lambda)$ apart from $L_\alpha(w^{-1} \cdot \lambda)$, namely $L_\alpha(s_\alpha w^{-1} \cdot \lambda)$. So the composition factors of Q_R^w are composition factors of $M_R(ws_\alpha w^{-1} \cdot \lambda) = M_R(s_{w(\alpha)} \cdot \lambda)$. Actually $Q_R^w = M_R^{ws}(s_{w(\alpha)} \cdot \lambda)$ in this case:

Lets consider the construction of the maps φ_i in the above. We start with the map $M_\alpha(\lambda) \rightarrow M_\alpha^s(\lambda)$ and then inflate to $M_{p_\alpha}(\lambda) \rightarrow M_{p_\alpha}^s(\lambda)$. In the case where q is not a root of unity it is easy to see that if $q^\rho \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ then this is an isomorphism and otherwise the kernel (and the cokernel) is isomorphic to $M_{p_\alpha}(s \cdot \lambda)$ which is a simple module. So after inducing we get the 4 term exact sequence

$$0 \rightarrow M_R(s \cdot \lambda) \rightarrow M_R(\lambda) \rightarrow M_R^s(\lambda) \rightarrow M_R^s(s \cdot \lambda) \rightarrow 0$$

since induction is exact on Verma modules. Use these observations on $w^{-1} \cdot \lambda$ and the fact that T_w is exact on Verma modules and we get a map $M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda)$ which is an isomorphism if $q^\rho \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ and otherwise we have the 4-term exact sequence

$$0 \rightarrow M_R^w(s \cdot \lambda) \rightarrow M_R^w(\lambda) \rightarrow M_R^{ws}(\lambda) \rightarrow M_R^{ws}(s \cdot \lambda) \rightarrow 0$$

Theorem 7.4 *Let R be a field (any characteristic) and let $q \in R$ be a non-root of unity. R is an A -algebra by sending v to q . Let $\lambda : U_q^0 \rightarrow R$ be an algebra homomorphism.*

$M_R(\lambda)$ has finite Jordan-Holder length and if $L_R(\mu)$ is a composition factor of $M_R(\lambda)$ then $\mu \uparrow \lambda$ where \uparrow is the usual strong linkage.

Proof. This will be proved by induction over \uparrow . If λ is anti-dominant (i.e. $q^\rho \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ for all $\alpha \in \Pi$) then we get that all the maps φ_i are isomorphisms and so $M_R(\lambda)$ is simple. Now assume λ is not anti-dominant. A composition factor $L_R(\mu)$ must be killed by one of the φ_i 's so must be a composition factor of Q_R^w for some w . By the above calculations we see that if $q^\rho \lambda(K_\alpha) \notin \pm q_\alpha^{\mathbb{Z}_{>0}}$ then $M_R^w(\lambda) \rightarrow M_R^{ws_\alpha}(\lambda)$ is an isomorphism and otherwise $Q_R^w = M_R^{ws_\alpha}(s_\alpha \cdot \lambda)$. By induction all the Verma modules with highest weight μ strongly linked to λ has finite length and the composition factors are strongly linked to μ . This finishes the induction. \square

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